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PUISEUX SERIES SOLUTIONS OF ODES

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Abstract. In this article, we will determine Puiseux series solutions of ordinary polynomial differential equations. We also study the binary complexity of computing such solutions. We will prove that this complexity bound is single exponential in the number of terms in the series. Our algorithm is based on a differential version of the Newton-Puiseux procedure for algebraic equations.

1. Introduction

Let $K = \mathbb{Q}(T_1, \dots, T_l)[\eta]$ be a finite extension of a finitely generated field over $\mathbb{Q}$. The variables $T_1, \dots, T_l$ are algebraically independent over $\mathbb{Q}$ and η is an algebraic element over the field $\mathbb{Q}(T_1, \dots, T_l)$ with the minimal polynomial $\phi \in \mathbb{Z}[T_1, \dots, T_l][Z]$. Let $\overline{K}$ be an algebraic closure of K and consider the two fields:

$$L = \cup_{\nu \in \mathbb{N}^*} K((x^{\frac{1}{\nu}})), \quad \mathcal{L} = \cup_{\nu \in \mathbb{N}^*} \overline{K}((x^{\frac{1}{\nu}}))$$

which are the fields of fraction-power series of x over K (respectively $\overline{K}$), i.e., the fields of Puiseux series of x with coefficients in K (respectively $\overline{K}$). Each element $\psi \in L$ (respectively $\psi \in \mathcal{L}$) can be represented in the form $\psi = \sum_{i \in \mathbb{Q}} c_i x^i$, $c_i \in K$ (respectively $c_i \in \overline{K}$). The order of ψ is defined by $\operatorname{ord}(\psi) := \min\{i \in \mathbb{Q}, c_i \neq 0\}$. The fields L and $\mathcal{L}$ are differential fields with the differential operator

$$\frac{d}{dx}(\psi) = \sum_{i \in \mathbb{Q}} i c_i x^{i-1}.$$

Let $F(y_0, \dots, y_n)$ be a polynomial in the variables $y_0, \dots, y_n$ with coefficients in L and consider the associated ordinary differential equation $F(y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx}) = 0$ which will be denoted by $F(y) = 0$. We will describe all the solutions of the differential equation $F(y) = 0$ in $\mathcal{L}$ by a differential version of the Newton polygon process. First write F in the form:

$$F = \sum_{i \in \mathbb{Q}, \alpha \in A} f_{i,\alpha} x^i y_0^{\alpha_0} \dots y_n^{\alpha_n}, \quad f_{i,\alpha} \in K$$

where $\alpha = (\alpha_0, \dots, \alpha_n)$ belongs to a finite subset A of $\mathbb{N}^{n+1}$. The order of F is defined by $\operatorname{ord}(F) := \min\{i \in \mathbb{Q}; f_{i,\alpha} \neq 0 \text{ for a certain } \alpha\}$. Without loss of

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generality we can suppose that each coefficient $f_{i,\alpha} \in \mathbb{Z}[T_1, \dots, T_l][\eta]$ and so it can be written in the form

$$f_{i,\alpha} = \sum_{j,j_1,\dots,j_l} b_{j,j_1,\dots,j_l} T_1^{j_1} \dots T_l^{j_l} \eta^j, \quad b_{j,j_1,\dots,j_l} \in \mathbb{Z}.$$

We define the degree of F with respect to x by $\deg_x(F) = \max\{i \in \mathbb{Q}; f_{i,\alpha} \neq 0 \text{ for a certain } \alpha\}$ (it can be equal to $+\infty$), the degree of F with respect to $T_1, \dots, T_l$ by $\deg_{T_1,\dots,T_l}(F) = \max\{\deg_{T_1,\dots,T_l}(f_{i,\alpha}); i \in \mathbb{Q}, \alpha \in A\}$. We denote by $l(b)$ the binary length of an integer b. The binary length of F is defined by $l(F) = \max\{l(f_{i,\alpha}); i \in \mathbb{Q}, \alpha \in A\}$ where $l(f_{i,\alpha})$ is the maximum of the binary lengths of its coefficients in $\mathbb{Z}$. We can define in the same manner the degrees and the binary length of ϕ. To estimate the binary complexity of the algorithm of this paper we suppose that $\deg_Z(\phi) \leq d_0$, $\deg_{T_1,\dots,T_l}(\phi) \leq d_1$, $l(\phi) \leq M_1$, $\deg_{y_0,\dots,y_n}(F) \leq d$, $\deg_{T_1,\dots,T_l}(F) \leq d_2$, $\deg_x(F) \leq d_3$ (d_3 can be equal to $+\infty$) and $l(F) \leq M_2$.

In this article, we will solve ordinary polynomial differential equations of the form $F(y) = 0$. For such an equation, we compute solutions in the set $\mathcal{L}$ of Puiseux series. There is no algorithm which decide whether a polynomial differential equation has Puiseux series as solutions. We get an algorithm which computes a finite extension of the ground field which generates the coefficients of the solutions. Algorithms which estimate the coefficients of the solutions are given in [10, 11].

To analyse the binary complexity of factoring ordinary linear differential operators, Grigoriev [8] describes an algorithm which computes a fundamental system of solutions of the *Riccatti* equation associated to an ordinary linear differential operator. The binary complexity of this algorithm is single exponential in the order n of the linear differential operator. There are also algorithms for computing series solutions with real exponents [9, 1, 6, 2] and complex exponents [6].

The article is organized as follows. In section 1, we state the main theorem. In section 2, we give a description of the Newton polygon associated to polynomial differential equations. The algorithm with its binary complexity analysis are described in section 3.

2. Main results

For each $i \in \mathbb{Q}$, let $R(i) = d_2(dd_0d_1)^{O(il)}$ and

$$S(i) = n^i d_2(dd_0d_1)^{O(il)}(M_1M_2)^{O(i)} \log_2^i(dd_3).$$

The main theorem of this article read as follows:

Theorem 2.1. *Let $F(y) = 0$ be a polynomial differential equation with the above bounds. There is an algorithm which computes all Puiseux series solutions of $F(y) = 0$ with coefficients in $\overline{K}$, i.e., all solutions of $F(y) = 0$ in $\mathcal{L}$. Namely, for each solution $\psi = \sum_{i\in\mathbb{Q}} c_i x^i \in \mathcal{L}$ of $F(y) = 0$, the algorithm computes an integer $\nu \in \mathbb{N}^*$ such that for each $i \in \mathbb{Q}$, it computes a finite extension $K_1 = K[\theta]$ of K where θ is an algebraic element over K computed with its minimal polynomial $\Phi \in K[Z]$ such that $\sum_{j\leq i, j\in\mathbb{Q}} c_j x^j \in K_1((x^{\frac{1}{\nu}}))$. Moreover, for any $j \leq i, j \in \mathbb{Q}$, we have the following bounds*

- $\deg_Z(\Phi) \leq d^i$.
- $\deg_{T_1,\dots,T_l}(\Phi)$, $\deg_{T_1,\dots,T_l}(c_j) \leq R(i)$.
- $l(\Phi), l(c_j) \leq S(i)$.
- *The binary complexity of this computation is $S(i)$.*

By [9, corollary of Lemma 3.1], the integer ν in Theorem 2.1 is constant, i.e., independent of i. This constant depends only on the solution ψ.

In general, we cannot compute a finite extension K_1 of K which contains all the coefficients of all the solutions of $F(y) = 0$ in $\mathcal{L}$ either an integer $\nu \in \mathbb{N}^*$ such that all the solutions of $F(y) = 0$ (in $\mathcal{L}$) are in $K_1((x^{\frac{1}{\nu}}))$. Namely, if we consider the polynomial

$$F(y_0, y_1, y_2) = xy_0y_2 - xy_1^2 + y_0y_1$$

then $\psi = cx^\mu$ is a solution of $F(y) = 0$ in $\mathcal{L}$ for all $c \in \mathbb{C}$ and all $\mu \in \mathbb{Q}$.

3. Newton polygons

Let F be a differential polynomial as in the introduction. We now define the Newton polygon of F. For every pair $(i, \alpha) \in \mathbb{Q} \times A$ such that $f_{i,\alpha} \neq 0$ (i.e., every existing term in F), we mark the point

$$P_{i,\alpha} := (i - \alpha_1 - 2\alpha_2 - \cdots - n\alpha_n, \alpha_0 + \alpha_1 + \cdots + \alpha_n) \in \mathbb{Q} \times \mathbb{N}.$$

We denote by $P(F)$ the set of all the points $P_{i,\alpha}$. The convex hull of these points and $(+\infty, 0)$ in the plane $\mathbb{R}^2$ is denoted by $\mathcal{N}(F)$ and is called the Newton polygon of the differential equation $F(y) = 0$ in the neighborhood of $x = 0$. If $\deg_{y_0,\dots,y_n}(F) = m$, then $\mathcal{N}(F)$ is located between the two lines $y = 0$ and $y = m$. For each $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$, we define the set

$$N(F, a, b) := \{(u, v) \in P(F),\ \forall (u', v') \in P(F), \quad au' + bv' \geq au + bv\}.$$

A point $P_{i,\alpha} \in P(F)$ is a vertex of the Newton polygon $\mathcal{N}(F)$ if there exist $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$ such that $N(F, a, b) = \{P_{i,\alpha}\}$. We remark that $\mathcal{N}(F)$ has a finite number of vertices. A pair of different vertices $e = (P_{i,\alpha}, P_{i',\alpha'})$ forms an edge of $\mathcal{N}(F)$ if there exist $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$ such that $e \subset N(F, a, b)$. We denote by $E(F)$ (respectively $V(F)$) the set of all the edges e (respectively all the vertices p) of $\mathcal{N}(F)$ for which $a > 0$ and $b \geq 0$ in the previous definitions. It is easy to prove that if $e \in E(F)$, then there exists a unique pair $(a(e), b(e)) \in \mathbb{Z}^2$ such that $\mathrm{GCD}(a(e), b(e)) = 1$, $a(e) > 0$, $b(e) \geq 0$ and $e \subset N(F, a(e), b(e))$ where GCD is an abbreviation of "Greatest Common Divisor". By the inclination of a line we mean the negative inverse of its geometric slope. If $e \in E(F)$, we can prove that the fraction $\mu_e := \frac{b(e)}{a(e)} \in \mathbb{Q}$ is the inclination of the straight line passing through the edge e. If $p \in V(F)$ and $N(F, a, b) = \{p\}$ for a certain (a, b), then the fraction $\mu := b/a \in \mathbb{Q}$ is the inclination of a straight line which intersects $\mathcal{N}(F)$ exactly in the vertex p.

For each $e \in E(F)$, we define the univariate polynomial (in a new variable C)

$$H_{(F,e)}(C) := \sum_{P_{i,\alpha} \in N(F, a(e), b(e))} f_{i,\alpha} C^{\alpha_0 + \alpha_1 + \cdots + \alpha_n} (\mu_e)_1^{\alpha_1} \dots (\mu_e)_n^{\alpha_n} \in K[C],$$

where $(\mu_e)_k := \mu_e(\mu_e - 1) \dots (\mu_e - k + 1)$ for any positive integer k. We call $H_{(F,e)}(C)$ the *characteristic* polynomial of F associated to the edge $e \in E(F)$. Its degree is at most $m = \deg_{y_0,\dots,y_n}(F) \leq d$.

If $\psi \in \mathcal{L}$ is a solution of the differential equation $F(y) = 0$ such that $\mathrm{ord}(\psi) = \mu_e$, i.e., ψ has the form $\psi = \sum_{i \in \mathbb{Q},\, i \geq \mu_e} c_i x^i$, $c_i \in \overline{K}$, then we have $H_{(F,e)}(c_{\mu_e}) = 0$, i.e. c_{μ_e} is a root of the polynomial $H_{(F,e)}$ in $\overline{K}$. This condition is called a necessary initial condition to have a solution of $F(y) = 0$ in the form of ψ (see [2, Lemma 1]). In fact, $H_{(F,e)}(c_{\mu_e})$ is equal to the coefficient of the lowest term in the expansion of

$F(\psi(x))$ with indeterminates μ_e and c_{μ_e}. Let $A_{(F,e)} := \{c \in \overline{K}, c \neq 0, H_{(F,e)}(c) = 0\}$.

For each $p = (u, v) \in V(F)$, let $\mu_1 < \mu_2$ be the inclinations of the adjacent edges at p in $\mathcal{N}(F)$, it is easy to prove that for all rational number $\mu = \frac{b}{a}$, $a \in \mathbb{N}^*$, $b \in \mathbb{N}$ such that $N(F, a, b) = \{p\}$, we have $\mu_1 < \mu < \mu_2$. We associate to p the polynomial

$$h_{(F,p)}(\mu) := \sum_{P_{i,\alpha}=p} f_{i,\alpha}(\mu)_1^{\alpha_1} \dots (\mu)_n^{\alpha_n} \in K[\mu],$$

which is called the *indicial* polynomial of F associated to the vertex p (here μ is considered as an indeterminate). Let $H_{(F,p)}(C) = C^v h_{(F,p)}(\mu)$ defined as above for edges $e \in E(F)$. Let $A_{(F,p)} := \{\mu \in \mathbb{Q},\ \mu_1 < \mu < \mu_2;\ h_{(F,p)}(\mu) = 0\}$.

Remark 3.1. Let $p = (u, v) \in V(F)$ and e be the edge of $\mathcal{N}(F)$ descending from p, then $h_{(F,p)}(\mu_e)$ is the coefficient of the monomial C^v in the expansion of the *characteristic* polynomial of F associated to e.

4. Differential version of the Newton-Puiseux algorithm

We describe now a differential version of the Newton-Puiseux algorithm to give formal Puiseux series solutions of the differential equation $F(y) = 0$. The input of the algorithm is a differential polynomial equation $F(y) = 0$ with the bounds described in the introduction. The algorithm will construct a tree $\mathcal{T}$ which depends only on F and on the field K. The root of $\mathcal{T}$ is denoted by τ_0. For each node τ of $\mathcal{T}$, it constructs the following elements:

- The field K_τ which is a finite extension of K.
- The primitive element θ_τ of the extension K_τ of K with its minimal polynomial $\phi_\tau \in K[Z]$.
- An element $c_\tau \in K_\tau$, a number $\mu_\tau \in \mathbb{Q} \cup \{-\infty, +\infty\}$ and an element $y_\tau = c_\tau x^{\mu_\tau} + y_{\tau_1} \in K_\tau((x^{\frac{1}{\nu(\tau)}}))$ where τ is a descendant of τ_1 (here $\mu_\tau > \mu_{\tau_1}$) and $\nu(\tau) \in \mathbb{N}^*$.
- The differential polynomial $F_\tau(y) = F(y+y_\tau)$ with coefficients in $K_\tau((x^{\frac{1}{\nu(\tau)}}))$.

We define the degree of τ by $\deg(\tau) = \mu_\tau \in \mathbb{Q}$, we have $\deg(\tau) = \deg_x(y_\tau)$ if $c_\tau \neq 0$. The level of the node τ, denoted by $lev(\tau)$, is the distance from τ_0 to τ.

For the root τ_0 we have $K_{\tau_0} = K$, $\theta_{\tau_0} = 1$, $\phi_{\tau_0} = Z - 1$, $c_{\tau_0} = y_{\tau_0} = 0$, $\deg(\tau_0) = \mu_{\tau_0} = -\infty$, $\nu(\tau_0) = 1$ and $F_{\tau_0}(y) = F(y)$.

A node τ of the tree $\mathcal{T}$ is a leaf of $\mathcal{T}$ if for each $e \in E(F_\tau)$ and for each $p \in V(F_\tau)$ we have $\mu_e \leq \deg(\tau)$ and $\mu_2 \leq \deg(\tau)$ and $y = 0$ is a solution of $F_\tau(y) = 0$, where $\mu_1 < \mu_2$ are the inclinations of the adjacent edges at p in $\mathcal{N}(F)$.

The algorithm constructs the tree $\mathcal{T}$ by induction on the level of its nodes. We suppose by induction on i that all the nodes of $\mathcal{T}$ of level $\leq i$ are constructed. Denote by $\mathcal{T}_i$ the set of these nodes. At the $(i+1)$-th step of the induction, for each node τ of level i which is not a leaf of $\mathcal{T}$ we consider the following two sets:

- $E'(F_\tau) = \{e \in E(F_\tau),\ \mu_e > \deg(\tau)\}$ and
- $V'(F_\tau) = \{p \in V(F_\tau),\ \mu_2 > \deg(\tau)\}$.

For each $e \in E'(F_\tau)$, compute a factorization of the polynomial $H_{(F_\tau,e)}(C) \in K_\tau[C]$ into irreducible factors over the field $K_\tau = K[\theta_\tau]$ in the form

$$H_{(F_\tau,e)}(C) = \lambda_e \prod_j H_j^{k_j}$$

where $0 \neq \lambda_e \in K_\tau$, $k_j \in \mathbb{N}^*$ and $H_j \in K_\tau[C]$ are monic and irreducibles over K_τ. We can do this factorization by the algorithm in [3, 4, 5, 7]. The elements of the set $A_{(F_\tau,e)}$ correspond to the roots of the factors $H_j \neq C$. We consider a root $c_j \in \overline{K}$ for each factor $H_j \neq C$ and we compute a primitive element $\theta_{j,e,\tau}$ of the finite extension $K_\tau[c_j] = K[\theta_\tau, c_j]$ of K with its minimal polynomial $\phi_{j,e,\tau} \in K[Z]$ using the algorithm in [3, 5, 7].

For each root c_j of $H_j \neq C$ we correspond a son σ of τ such that $\theta_\sigma = \theta_{j,e,\tau}$, the field $K_\sigma = K[\theta_{j,e,\tau}]$ and the minimal polynomial of θ_σ over K is $\phi_\sigma = \phi_{j,e,\tau}$. Moreover, $c_\sigma = c_j$, $\mu_\sigma = \mu_e$, $y_\sigma = c_\sigma x^{\mu_\sigma} + y_\tau$ and $F_\sigma(y) = F(y + y_\sigma)$. For $\nu(\sigma)$, we take $\nu(\sigma) = LCM(\nu(\tau), a(e))$ for example.

For each $p \in V'(F_\tau)$, we consider the *indicial* polynomial $h_{(F_\tau,p)}(\mu) \in K_\tau[\mu]$ of F_τ associated to p. To each $\mu \in A_{(F_\tau,p)}$ such that $\mu > \deg(\tau)$ and $0 \neq c \in \overline{K}$ (where c is given by its minimal polynomial over K), we correspond a son σ of τ such that $\theta_\sigma = c$, $c_\sigma = c$, $\mu_\sigma = \mu$. This completes the description of all the sons of the node τ of the tree $\mathcal{T}$.

Remark 4.1. (i) If ($E'(F_\tau) \neq \emptyset$ or $V'(F_\tau) \neq \emptyset$) and $y = 0$ is a solution of $F_\tau(y) = 0$ then one of the sons of τ is a leaf σ for which $F_\sigma = F_\tau$, $\mu_\sigma = +\infty$ and $c_\sigma = 0$.

(ii) For any node τ of $\mathcal{T}$ such that $\deg(\tau) \neq \infty$, if $y = 0$ is not a solution of $F_\tau(y) = 0$ then $E'(F_\tau) \neq \emptyset$.

Let $\mathcal{U}$ be the set of all the vertices τ of $\mathcal{T}$ such that either $\deg(\tau) = +\infty$ and for the ancestor τ_1 of τ it holds $\deg(\tau_1) < +\infty$ or $\deg(\tau) < +\infty$ and τ is a leaf of $\mathcal{T}$. For each $\tau \in \mathcal{U}$, there exists a sequence $(\tau_i(\tau))_{i\geq 0}$ of vertices of $\mathcal{T}$ such that $\tau_0(\tau) = \tau_0$ and $\tau_{i+1}(\tau)$ is a son of $\tau_i(\tau)$ for all $i \geq 0$. For each $\tau \in \mathcal{U}$, the element

$$y_\tau = \sum_{i\geq 0} c_{\tau_i(\tau)} x^{\mu_{\tau_i(\tau)}} \in K_\tau((x^{\frac{1}{\nu(\tau)}}))$$

is a solution of $F(y) = 0$. In fact, there are two possibilities to τ: if $\deg(\tau) = +\infty$ then $y = 0$ is a solution of $F_{\tau_1}(y) = 0$ where τ is a son of τ_1 and so y_{τ_1} is a solution of $F(y) = 0$. If $\deg(\tau) < +\infty$ and τ is a leaf of $\mathcal{T}$ then $y = 0$ is a solution of $F_\tau(y) = F(y + y_\tau) = 0$ and so y_τ is a solution of $F(y) = 0$. This defines a bijection between $\mathcal{U}$ and the set of the solutions of $F(y) = 0$ in $\mathcal{L}$.

To analyse the binary complexity of the algorithm, we begin by estimating the binary complexity of computing all the sons of the root τ_0 of $\mathcal{T}$. For each $e \in E(F)$, we consider the polynomial $H_{(F,e)}(C) \in K[C]$, its degree with respect to C (respectively $T_1, \dots, T_l$) is bounded by d (respectively d_2). We have $\mu_e \leq \frac{d_3}{d}$ and its binary length is $l(\mu_e) \leq O(\log_2(dd_3))$ (using the fact that μ_e is the inclination of the straight line passing through e). Then the binary length of $H_{(F,e)}(C)$ is bounded by $M_2 + ndO(\log_2(dd_3))$. By the algorithm of [3, 4, 5, 7] the binary complexity of factoring $H_{(F,e)}(C)$ into irreducible polynomials over K is

$$(dd_1d_2)^{O(l)}(nd_0M_1M_2\log_2(dd_3))^{O(1)}.$$

Moreover, each factor $H_j \in K[C]$ of $H_{(F,e)}(C)$ satisfies the following bounds (see [5, Lemma 1.3]): $\deg_C(H_j) \leq d$, $\deg_{T_1,\dots,T_l}(H_j) \leq d_2(dd_0d_1)^{O(1)}$ and

$$l(H_j) \leq nld_2(dd_0d_1)^{O(1)}M_1M_2\log_2(dd_3).$$

By the induction we suppose that the following bounds hold at the i-th step of the algorithm for each node τ of $\mathcal{T}$ of level i:

- $\deg_Z(\phi_\tau) \le d^i$.
- $\deg_{T_1,\dots,T_l}(\phi_\tau)$, $\deg_{T_1,\dots,T_l}(c_\tau) \le R(i)$.
- $l(\phi_\tau)$, $l(c_\tau) \le S(i)$ where $R(i)$ and $S(i)$ are as in the introduction.
- $\mu_\tau \le i(\frac{d_3}{d})$ and then $l(\mu_\tau) \le O(\log_2(idd_3))$.

Then we have the following bounds for the differential polynomial $F_\tau(y) = F(y + y_\tau) \in K_\tau((x^{\frac{1}{\nu(\tau)}}))[y_0,\dots,y_n]$:

- $\deg_{y_0,\dots,y_n}(F_\tau) \le d$.
- $\deg_{T_1,\dots,T_l}(F_\tau) \le d_2 + d\deg_{T_1,\dots,T_l}(c_\tau) \le R(i)$.
- $\deg_x(F_\tau) \le d_3 + d\mu_\tau \le (i+1)d_3$.
- $l(F_\tau) \le M_2 + dl(c_\tau) \le S(i)$.

We compute a primitive element η_1 of the finite extension K_τ over the field $\mathbb{Q}(T_1,\dots,T_l)$, i.e., $K_\tau = K[\theta_\tau] = \mathbb{Q}(T_1,\dots,T_l)[\eta][\theta_\tau] = \mathbb{Q}(T_1,\dots,T_l)[\eta_1]$ by [8, corollary of Proposition 1.4] (see also [5, section 3 chapter 1]). Moreover, $\eta_1 = \eta + \gamma\theta_\tau$ where $0 \le \gamma \le [K_\tau : \mathbb{Q}(T_1,\dots,T_l)] = \deg_Z(\phi)\deg_Z(\phi_\tau) \le d^i d_0$ and we can compute the monic minimal polynomial $\phi_1 \in \mathbb{Q}(T_1,\dots,T_l)[Z]$ of η_1 which satisfies the following bounds:

- $\deg_Z(\phi_1) \le d^i d_0$
- $\deg_{T_1,\dots,T_l}(\phi_1) \le (d^i d_0)^{O(1)}$
- $l(\phi_1) \le S(i)$.
- This computation can be done with binary complexity $S(i)$.

For each $e \in E'(F_\tau)$, we consider the polynomial $H_{(F_\tau,e)}(C) \in K_\tau[C]$, its degree with respect to C (respectively $T_1,\dots,T_l$) is bounded by d (respectively $R(i)$). We have $\mu_e \le (i+1)(\frac{d_3}{d})$ and its binary length is $l(\mu_e) \le O\Big(\log_2((i+1)dd_3)\Big)$. Then the binary length of $H_{(F_\tau,e)}(C)$ is bounded by

$$l(F_\tau) + ndl(\mu_e) \le S(i).$$

By the algorithm of [3, 4, 5, 7] the binary complexity of factoring $H_{(F_\tau,e)}(C)$ into irreducible polynomials over $K_\tau = \mathbb{Q}(T_1,\dots,T_l)[\eta_1]$ is $S(i)$. Moreover, each factor $H_j \in K_\tau[C]$ of $H_{(F_\tau,e)}(C)$ satisfies the following bounds:

- $\deg_C(H_j) \le d$
- $\deg_{T_1,\dots,T_l}(H_j) \le R(i)$.
- $l(H_j) \le S(i)$.

Let $c_j \in \overline{K}$ be a root of H_j. We can compute by the corollary of Proposition 1.4 of [8] a primitive element $\theta_{j,e,\tau}$ of the finite extension $K_\tau[c_j] = K[\theta_\tau, c_j]$ of K with its minimal polynomial $\phi_{j,e,\tau} \in K[Z]$. We can express $\theta_\sigma = \theta_{j,e,\tau}$ in the form $\theta_\sigma = \theta_\tau + \gamma_j c_j$ where $0 \le \gamma_j \le \deg_Z(\phi_\tau)\deg_C(H_j) \le d^{i+1}$ and $c_\sigma = c_j$ in the form

$$c_\sigma = \sum_{0 \le t < d^{i+1}} b_t \theta_\sigma^t$$

where $b_t \in K$. Moreover, the following bounds hold:

- $\deg_Z(\phi_\sigma) \le d^{i+1}$.
- $\deg_{T_1,\dots,T_l}(\phi_\sigma)$, $\deg_{T_1,\dots,T_l}(b_t) \le R(i)$.
- $l(\phi_\sigma)$, $l(b_t) \le S(i)$.
- $\mu_\sigma = \mu_e \le (i+1)(\frac{d_3}{d})$ and then $l(\mu_\sigma) \le O\Big(\log_2((i+1)dd_3)\Big)$.

This computation can be done with binary complexity $S(i)$ and thus the total binary complexity of computing all the sons σ of τ is $S(i)$.

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