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Solvability of Some Perturbed Sequence Spaces Equations with Operators

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Abstract. In this paper, we apply the results stated in [19] to the solvability of the sequence spaces equations (SSE) $\mathcal{E}+F_x = F_b$, where $\mathcal{E}, F$ are linear spaces of sequences and b, x are positive sequences (x is the unknown). In this way, we solve the (SSE) of the form $(E_a)_{G(\alpha,\beta)} + F_x = F_b$, where $G(\alpha,\beta)$ is a factorable triangle matrix defined by $[G(\alpha,\beta)]_{nk} = \alpha_n\beta_k$ for $k \leq n$ and $(E,F) \in \{(\ell_\infty, c), (c_0, \ell_\infty), (c_0, c), (\ell^p, c), (\ell^p, \ell_\infty), (w_0, \ell_\infty)\}$ with $p \geq 1$. Then we deal with some (SSE) involving the matrices $C(\lambda)$, C_1 and $\overline{N}_q$. Finally, we solve the (SSE) with operator of the form $(E_a)_{\Sigma^2} + F_x = F_b$.

1. Introduction

Let ω be the set of all complex sequences $y = (y_n)_{n\geq 1}$. Let U (resp. U^+) be the set of sequences y such that $y_n \neq 0$ (resp. $y_n > 0$) for all $n \geq 1$. Denote by ℓ_∞, c and c_0 the sets of all bounded, convergent and null sequences, respectively. For $1 \leq p < \infty$, denote by ℓ^p the set of sequences $y \in \omega$ such that $\sum_{n=1}^{\infty} |y_n|^p < \infty$. For $y \in \omega$ and $a \in U$, define the sequence $y/a = (y_n/a_n)_{n\geq 1}$. In particular, $1/a = e/a$, where $e = \mathbf{1}$ is the sequence with $e_n = 1$ for all n. For $a \in U^+$ and E is any subset of ω, we put:

$$E_a = \{y \in \omega : y/a \in E\}.$$

In [2], the sets s_a, s_a^0 and $s_a^{(c)}$ were defined for $a \in U^+$ by E_a and $E = \ell_\infty, c_0, c$, respectively. In [3], the sum $E_a + F_b$ and the product $E_a * F_b$ were defined where E, F are any of the symbols s, s^0, or $s^{(c)}$. In [6], sequences spaces inclusion equations (SSIE) of the form $G_b \subset E_a + F_x$ are solved, where $a, b \in U^+$ and E, F, $G \in \{s^0, s^{(c)}, s\}$ and some applications were given to sequence spaces inclusions equations with operators. Recall that the spaces w_∞ and w_0 of strongly bounded and summable sequences are the sets of all y such that $\left(n^{-1}\sum_{k=1}^{n} |y_k|\right)_n$ is bounded and tends to zero respectively. These spaces were studied by Maddox [23]

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and Malkowsky, Rakočević [22]. In [9, 14] were given some properties of well known operators defined by the sets $W_a = (w_\infty)_a$ and $W_a^0 = (w_0)_a$.

In this paper, we deal with special sequence spaces equations (SSE), which are determined by an identity, for which each term is a *sum* or *a sum of products of sets of the form* $(E_a)_T$ *and* $\left(E_{f(x)}\right)_T$ where f maps U^+ to itself, E is any linear space of sequences and T is a triangle. Some results on (SSE) and (SSIE) were stated in [4–8, 12, 15, 16, 18, 20]. In [6], we dealt with the (SSIE) with operators $E_a + (F_x)_\Delta \subset s_x^{(c)}$ where E and F are any of the sets c_0, c, or s_1. In [15], we determined the set of all positive sequences x for which the *(SSIE)* $\left(s_x^{(c)}\right)_{B(r,s)} \subset \left(s_x^{(c)}\right)_{B(r',s')}$ holds, where r, r', s' and s are real numbers, and $B(r,s)$ is the generalized operator of the first difference defined by $(B(r,s)y)_n = ry_n + sy_{n-1}$ for all $n \geq 2$ and $(B(r,s)y)_1 = ry_1$. In this way, we determined the set of all positive sequences x for which $(ry_n + sy_{n-1})/x_n \to l$ implies $(r'y_n + s'y_{n-1})/x_n \to l'$ $(n \to \infty)$ for all y and for some scalars l and l'. In [8], we used the sets of analytic and entire sequences denoted by $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ and defined by $\sup_{n\geq 1}\left(|y_n|^{1/n}\right) < \infty$ and $\lim_{n\to\infty}\left(|y_n|^{1/n}\right) = 0$, respectively. Then we dealt with a class of (SSE) with operators of the form $E_T + F_x = F_b$, where T is either Δ or Σ and E is any of the sets c_0, c, ℓ_∞, ℓ_p, $(p \geq 1)$, w_0, $\mathbf{\Gamma}$, or $\mathbf{\Lambda}$ and $F = c$, ℓ_∞ or $\mathbf{\Lambda}$. In [11], we solved the (SSE) defined by $(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$ where E is either c_0, or ℓ^p, and the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ where E is either c, or ℓ_∞. In [10, 13] we dealt with the sequence spaces inclusion equations (SSIE) defined by $F_b \subset E_a + F'_x$ where a and b are positive sequences and E, F and F' are linear subspaces of ω and we solved the (SSE) defined by $E_a + F_x = F_b$ when $e \notin F$.

In [19], we extended some of the results stated in [8] and we solved a new class of (SSE) of the form $(E_a)_T + F_x = F_b$ where $(E, F) \in \Theta$ and

$$\Theta = \left\{(\ell_\infty, c), (c_0, \ell_\infty), (c_0, c), (\ell^p, c), (\ell^p, \ell_\infty), (w_0, \ell_\infty)\right\},$$

with $p \geq 1$ and T is a triangle whose the inverse has finite columns.

The paper is organized as follows. In Section 2, we recall the notions of sequence spaces, matrix transformations and triangles. Then, we define the set W_a^0 for $a \in U^+$. We also recall some results on matrix transformations [1] and sequence spaces equations studied in [19]. In Section 3, we deal with the solvability of some *perturbed equations* of the form $(E_a)_{G(\alpha,\beta)} + F_x = F_b$, where $a, b, \alpha, \beta \in U^+$, $(E, F) \in \Theta$ and $G(\alpha, \beta) = D_\alpha \Sigma D_\beta$ is a factorable matrix. Then we deal with the (SSE) involving some factorable matrices of the form $C(\lambda)$, C_1 or $\overline{N}_q$. Finally, in Section 4, we apply the previous results to the (SSE) involving the operator Σ^2.

2. Preliminaries and previous results

For any given infinite matrix $A = (\mathbf{a}_{nk})_{n,k\geq 1}$, we define the operators $A_n = (\mathbf{a}_{nk})_{k\geq 1}$ for any integer $n \geq 1$, by $A_n y = \sum_{k=1}^{\infty} \mathbf{a}_{nk} y_k$, where $y = (y_n)_{n\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the operator A defined by $Ay = (A_n y)_{n\geq 1}$ mapping between sequence spaces. When A maps E into F, where E and F are subsets of ω, we write $A \in (E, F)$, (cf. [23, 24]). For any subset F of ω, the domain of A in F is defined by:

$$F_A = \{y \in \omega : Ay \in F\}.$$

For $a = (a_n)_{n\geq 1} \in \omega$ we define the diagonal matrix D_a by $[D_a]_{nn} = a_n$ for all n. Let E be any subset of ω and $a \in U^+$, then

$$E_a = D_a * E = \{y \in \omega : y/a \in E\}.$$

For $a \in U^+$, we define the sets s_a, s_a^0, $s_a^{(c)}$ and ℓ_a^p (for $p \geq 1$) as follows (cf. [2]):

$$s_a = D_a * \ell_\infty, \quad s_a^0 = D_a * c_0, \quad s_a^{(c)} = D_a * c, \quad \ell_a^p = D_a * \ell^p.$$

For $a = (R^n)_{n\geq 1}$ with $R > 0$, we write s_R, s_R^0, $s_R^{(c)}$, and ℓ_R^p for the sets s_a, s_a^0, $s_a^{(c)}$ and ℓ_a^p, respectively. When $R = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$ and $s_1^{(c)} = c$. Recall that

$$(c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty) = S_1,$$

where S_1 is the set of infinite matrices $A = (\mathbf{a}_{nk})_{n,k\geq 1}$ satisfying:

$$\sup_n \left(\sum_{k=1}^{\infty} |\mathbf{a}_{nk}| \right) < \infty. \tag{1}$$

For $a \in U^+$, we define the set of sequences that are *a-strongly convergent to zero* by

$$W_a^0 = (w_0)_a = \left\{ y \in \omega : \lim_{n\to\infty} \left(n^{-1} \sum_{k=1}^{n} |y_k| / a_k \right) = 0 \right\},$$

(cf. [9, 14, 17]). If $a = (r^n)_{n\geq 1}$, then, the set W_a^0 is denoted by W_r^0. For $r = 1$, we obtain the well known set

$$w_0 = \left\{ y \in \omega : \lim_{n\to\infty} \left(n^{-1} \sum_{k=1}^{n} |y_k| \right) = 0 \right\}$$

called the *space of sequences that are strongly summable to zero by the Cesàro method* (cf. [21]). We recall some results that are direct consequence of [[1], Theorem 2.4], where we let $\sigma = (\sigma_n)_{n\geq 1}$ with

$$\sigma_n = \sigma_n(A) = \sum_{\nu=0}^{\infty} 2^\nu \max_{2^\nu \leq k \leq 2^{\nu+1}-1} |\mathbf{a}_{nk}|, \tag{2}$$

for $A = (\mathbf{a}_{nk})_{n,k\geq 1}$. From [1] we obtain the following lemma.

Lemma 2.1.

i) We have $A \in (w_0, \ell_\infty)$ if and only if

$$\sigma \in \ell_\infty. \tag{3}$$

ii) $A \in (w_0, c_0)$ if and only if (3) holds and $\lim_{n\to\infty} \mathbf{a}_{nk} = 0$ for all k.

2.1. Triangles

We call *triangle* every infinite matrix $T = (t_{nk})_{n,k\geq 1}$ such that $t_{nk} = 0$ for all $n < k$ and $t_{nn} \neq 0$ for all $n \geq 1$. For any $\lambda \in U$, let $\Delta(\lambda)$ and $C(\lambda)$ be the triangles defined by (see for instance [2]):

$$[\Delta(\lambda)]_{nk} = \begin{cases} \lambda_n & \text{for } k = n \\ -\lambda_{n-1} & \text{for } k = n-1 \\ 0 & \text{for } k \neq n-1 \text{ and } k \neq n \ (n \geq 1) \end{cases}$$

and

$$[C(\lambda)]_{n,k} = \begin{cases} 1/\lambda_n & \text{for } k \leq n \\ 0 & \text{otherwise} \end{cases}$$

Note that $C(\lambda)$ is the inverse of $\Delta(\lambda)$. Let $e \in U$, defined by $e_n = 1$ for all $n \geq 1$. In all that follows we use the convention $a_n = 0$ for all $n \leq 0$. Then, $\Delta = \Delta(e)$ is the well known operator of the first-difference defined by

$$\Delta_n x = x_n - x_{n-1} \text{ for all } n \geq 1.$$

Recall that the operator Δ is invertible and its inverse is usually written $\Sigma = C(e)$. For any $\lambda \in U$, we have $\Delta(\lambda) = \Delta D_\lambda$ and its inverse is determined by $C(\lambda) = (\Delta(\lambda))^{-1} = D_{1/\lambda}\Sigma$.

2.2. Solvability of the (SSE) $(E_a)_T + F_x = F_b$ *where* $F \in \{c, \ell_\infty\}$

In this part we recall some results stated in [19].

Theorem 2.2. *[[19], Theorem 5.1 p. 5128]*
Let $a, b \in U^+$, *T be a triangle and let* $(E, F) \in \Theta$. *Assume for every positive integer k, there is an integer* $i_k > k$ *such that*

$$T_{nk}^{-1} = 0 \text{ for all } n \geq i_k. \tag{4}$$

Then the set of all the solutions of the (SSE):

$$(E_a)_T + F_x = F_b$$

is given by:

$$\mathbf{S}_T = \begin{cases} cl^F(b), & \text{if } D_{1/b}T^{-1}D_a \in (E, F), \\ \emptyset, & \text{if } D_{1/b}T^{-1}D_a \notin (E, F). \end{cases}$$

To state the next results, we use the sequence σ defined in Lemma 2.1 where A is a triangle L, so we obtain

$$\sigma_n = \sigma_n(L) = \sum_{\nu=0}^{\nu_n - 1} 2^\nu \max_{2^\nu \leq k \leq 2^{\nu+1}-1} |L_{nk}| + 2^{\nu_n} \max_{2^{\nu_n} \leq k \leq n} |L_{nk}| \tag{5}$$

where for every n, ν_n is an integer uniquely defined by $2^{\nu_n} \leq n \leq 2^{\nu_n+1} - 1$. In the following we use the next conditions where T is a triangle

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} \left|T_{nk}^{-1}\right| a_k = 0, \tag{6}$$

$$\sup_n \left(\frac{1}{b_n} \sum_{k=1}^{n} \left|T_{nk}^{-1}\right| a_k \right) < \infty, \tag{7}$$

$$\sup_n \left(\frac{1}{b_n^q} \sum_{k=1}^{n} \left|T_{nk}^{-1}\right|^q a_k^q \right) < \infty \text{ with } q = p/(p-1), (p > 1), \tag{8}$$

and

$$\sup_{n \geq k} \left(\frac{1}{b_n} \left|T_{nk}^{-1}\right| a_k \right) < \infty. \tag{9}$$

Theorem 2.3. *[[19], Corollary 5.2 p. 5129]*
Let $a, b \in U^+$ *and let T be a triangle such that the condition in (4) holds. Then we have:*

i) *a) The solutions of the equation* $(s_a)_T + s_x^{(c)} = s_b^{(c)}$ *are determined by*

$$S_\infty^c(T) = \begin{cases} cl^c(b), & \text{if (6) holds,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

b) The solutions of the equation $\left(s_a^0\right)_T + F_x = F_b$ *with* $F = c$, *or* ℓ_∞ *are determined by*

$$S_0^F(T) = \begin{cases} cl^F(b), & \text{if (7) holds,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

ii) Let $F = c$, *or* ℓ_∞. *Then the solutions of the equation* $\left(\ell_a^p\right)_T + F_x = F_b$ *with* $p \geq 1$ *are determined in the following way.*

a) If $p > 1$, *then*

$$S_p^F(T) = \begin{cases} cl^F(b), \text{ if (8) holds,} \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

b) If $p = 1$, *then*

$$S_1^F(T) = \begin{cases} cl^F(b), \text{ if (9) holds,} \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

iii) The solutions of the equation $\left(W_a^0\right)_T + s_x = s_b$ *are determined by*

$$S_{w_0}^\infty(T) = \begin{cases} cl^\infty(b), \text{ if } \sup_n \left\{\sigma_n\left(D_{1/b}T^{-1}D_a\right)\right\} < \infty, \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

3. Solvability of some (SSE) that involve the factorable matrix

In this section, we deal with the solvability of the (SSE):

$$(E_a)_{G(\alpha,\beta)} + F_x = F_b$$

where $a, b, \alpha, \beta \in U^+$ and $(E, F) \in \Theta$. Then we apply these results to the (SSE) involving the matrices $C(\lambda)$, C_1 or $\overline{N}_q$. Here, we consider the factorable matrix defined, for $\alpha, \beta \in U^+$, by

$$G(\alpha, \beta) = D_\alpha \Sigma D_\beta.$$

The matrix $G(\alpha, \beta)$ is the triangle defined by $[G(\alpha, \beta)]_{nk} = \alpha_n \beta_k$ for $k \leq n$, for all n.

3.1. On the (SSE) of the form $(E_a)_{G(\alpha,\beta)} + F_x = F_b$ *where* $(E, F) \in \Theta$.

In this part, among other things we determine the set $S_\infty^c(G)$ of all positive sequences x that satisfy $(s_a)_{G(\alpha,\beta)} + s_x^{(c)} = s_b^{(c)}$. This set is associated with the next statement. For every $y \in \omega$, we have $y_n/b_n \to L_1$ $(n \to \infty)$ if and only if there are two sequences u, v with $y = u + v$ such that

$$\sup_n \left(\frac{\alpha_n}{a_n}\left|\sum_{k=1}^n \beta_k u_k\right|\right) < \infty \text{ and } \frac{v_n}{x_n} \to L_2 \ (n \to \infty)$$

for some scalars L_1 and L_2. Similarly, the set $S_p^\infty(G)$, $(p > 1)$, is associated with the equation $\left(\ell_a^p\right)_{G(\alpha,\beta)} + s_x = s_b$ and with the next statement. The condition $\left|y_n\right|/b_n \leq K_1$ holds if and only if there are two sequences u, v with $y = u + v$ such that $\sum_{n=1}^\infty \left|\alpha_n a_n^{-1}\left(\sum_{k=1}^n \beta_k u_k\right)\right|^p < \infty$ and $|v_n|/x_n \leq K_2$ for all n, for all y and for some scalars $K_1, K_2 > 0$. To state the next results, we let

$$\gamma = a/\alpha \quad \text{and } \tau_n = (\gamma_n + \gamma_{n-1})/(b_n\beta_n).$$

We obtain the following result.

Theorem 3.1. *Let $a, b, \alpha, \beta \in U^{+}$. Then we have:*

i) The solutions of the (SSE) $(s_a)_{G(\alpha,\beta)} + s_x^{(c)} = s_b^{(c)}$ are determined by

$$S_{\infty}^{c}(G) = \begin{cases} cl^{c}(b), \text{ if } \lim_{n\to\infty} \tau_n = 0, \\ \emptyset, \quad \text{otherwise.} \end{cases}$$

ii) Let $F = c$, or ℓ_∞. Then the solutions of the (SSE) $\left(\ell_a^{p}\right)_{G(\alpha,\beta)} + F_x = F_b$ with $p \geq 1$ and $\left(s_a^{0}\right)_{G(\alpha,\beta)} + F_x = F_b$ satisfy $S_p^F(G) = S_0^F(G)$ and are determined by

$$S_p^{F}(G) = \begin{cases} cl^{F}(b), \text{ if } \sup_{n\geq 1} \tau_n < \infty, \\ \emptyset, \quad \text{otherwise.} \end{cases}$$

iii) The solutions of the (SSE) defined by $\left(W_a^{0}\right)_{G(\alpha,\beta)} + s_x = s_b$ are determined by

$$S_{w_0}^{\infty}(G) = \begin{cases} cl^{\infty}(b), \text{ if } \sup_{n\geq 1}(n\tau_n) < \infty, \\ \emptyset, \quad \text{otherwise.} \end{cases}$$

Proof. Remark that $G^{-1}(\alpha,\beta)$ satisfies the condition in (4). Indeed, the matrix $G^{-1}(\alpha,\beta)$ is the triangle defined by

$$G^{-1}(\alpha,\beta) = \left(D_\alpha \Sigma D_\beta\right)^{-1} = D_{1/\beta}\Delta D_{1/\alpha},$$

whose the nonzero entries are given by

$$\left[G^{-1}(\alpha,\beta)\right]_{nn} = 1/(\alpha_n\beta_n) \text{ and } \left[G^{-1}(\alpha,\beta)\right]_{n,n-1} = -1/(\alpha_{n-1}\beta_n)$$

for all $n \geq 1$ with $\left[G^{-1}(\alpha,\beta)\right]_{1,0} = 0$. Then we have

$$D_{1/b}G^{-1}(\alpha,\beta) D_a = D_{1/(b\beta)}\Delta D_{a/\alpha},$$

that is,

$$\left[D_{1/b}G^{-1}(\alpha,\beta) D_a\right]_{nn} = \gamma_n (\beta_n b_n)^{-1} \text{ and } \left[D_{1/b}G^{-1}(\alpha,\beta) D_a\right]_{n,n-1} = -\gamma_{n-1}(\beta_n b_n)^{-1}$$

for all $n \geq 1$ with $\left[D_{1/b}G^{-1}(\alpha,\beta) D_a\right]_{1,0} = 0$. Trivially we have

$$\lim_{n\to\infty} \left[D_{1/b}G^{-1}(\alpha,\beta) D_a\right]_{nk} = 0 \text{ for all } k \geq 1. \tag{10}$$

i) follows from Theorem 2.3 where $T = G(\alpha,\beta)$ and the condition in (6) is equivalent to $\lim_{n\to\infty} \tau_n = 0$.

ii) The case of $S_0^F(G)$ follows from Theorem 2.3 where the condition in (7) is equivalent to $\sup_n \tau_n < \infty$. Now, we study the case of the set $S_p^F(G)$.

- Case $p > 1$. Here, the condition in (8) is equivalent to

$$\sup_{n\geq 1} \left\{ \left(\frac{\gamma_{n-1}}{b_n\beta_n}\right)^{q} + \left(\frac{\gamma_n}{b_n\beta_n}\right)^{q} \right\} < \infty \text{ with } q = p/(p-1). \tag{11}$$

  It can easily be seen that the condition in (11) holds if and only if $\sup_n \tau_n < \infty$, and we conclude by Theorem 2.3.

- The case $p = 1$ follows from the equivalence of the condition in (9) and the conditions

$$\left(\gamma_{n-1} b_n^{-1} \beta_n^{-1}\right)_{n \geq 1} \in \ell_\infty \quad \text{and} \quad \gamma/(b\beta) \in \ell_\infty.$$

But these two last conditions are equivalent to $\sup_n \tau_n < \infty$ and we conclude by Theorem 2.3. This completes the proof of ii).

iii) Here we need to simplify the sum σ_n defined in (5) with $L = D_{1/b} G^{-1}(\alpha, \beta) D_a$. Let ν_n be the integer defined by $2^{\nu_n} \leq n \leq 2^{\nu_n + 1} - 1$ for all n. If $2^{\nu_n} = n$, then we obtain

$$\sigma_n = \sigma_n^{(1)} = 2^{\nu_n - 1} \left| \left[D_{1/b} G^{-1}(\alpha, \beta) D_a \right]_{n,n-1} \right| + 2^{\nu_n} \left[D_{1/b} G^{-1}(\alpha, \beta) D_a \right]_{n,n}$$

and

$$\sigma_n = \sigma_n^{(1)} = \frac{n}{b_n \beta_n} \left(\frac{1}{2} \gamma_{n-1} + \gamma_n \right).$$

Now, let

$$\mu_n = \max\left(\gamma_{n-1}/b_n\beta_n, \gamma_n/b_n\beta_n\right).$$

Then if $2^{\nu_n} = n - 1$, we have $\sigma_n = \sigma_n^{(2)} = (n-1)\mu_n$. If $2^{\nu_n} < n - 1$, then we have $\sigma_n = \sigma_n^{(3)} = 2^{\nu_n}\mu_n$. Since we have $\tau_n/2 \leq \mu_n \leq \tau_n$ and $(n+1)/2 \leq 2^{\nu_n} \leq n$, there is $K > 0$ such that $Kn\tau_n \leq \sigma_n^{(i)} \leq n\tau_n$ for $i = 1, 2, 3$ and for all n. Then it can easily be seen that $\left(\sigma_n^{(i)}\right)_{n\geq 1} \in \ell_\infty$ for $i = 1, 2, 3$ if and only if $(n\tau_n)_{n \geq 1} \in \ell_\infty$. We conclude by Lemma 2.1 and Theorem 2.2. This completes the proof of the theorem. □

In all that follows we write $E^+ = E \cap U^+$ for any given subset E of ω. The previous results lead to the next remarks.

Remark 3.2. *Let F be either c, or ℓ_∞ and for any given $b \in U^+$, let $\mathcal{A}_b^p$, (resp. $\mathcal{A}_b^0$) be the set of all sequences $a \in U^+$ such that the (SSE) $F_x = F_b$ and the perturbed equation $\left(\ell_a^p\right)_\Sigma + F_x = F_b$ with $p \geq 1$, (resp. $\left(s_a^0\right)_\Sigma + F_x = F_b$) have the same solutions. By Theorem 3.1 ii), since $\tau_n = (a_n + a_{n-1}) b_n^{-1}$, we obtain*

$$\mathcal{A}_b^p = \mathcal{A}_b^0 = s_b^+ \cap s^+_{(b_{n+1})_{n\geq 1}} = s^+_{(\max(b_n, b_{n+1}))_{n \geq 1}}.$$

Remark 3.3. *Let $b \in U^+$ and let $\mathcal{A}_b^\sigma$ be the set of all sequences $a \in U^+$ such that the (SSE) $s_x = s_b$ and the perturbed equation $\left(W_a^0\right)_\Sigma + s_x = s_b$ have the same set of solutions. By Theorem 3.1 iii), it can easily be seen that*

$$\mathcal{A}_b^\sigma = s^+_{(b_n/n)_{n\geq 1}} \cap s^+_{(b_{n+1}/(n+1))_{n \geq 1}}.$$

In the case when a/α is monotone, we obtain the next results.

Corollary 3.4. *Let $a, b, \alpha, \beta \in U^+$ and assume that either $\gamma = a/\alpha$, or $b\beta$ is a nondecreasing sequence. Then we have:*

i) The solutions of the (SSE) $(s_a)_{G(\alpha,\beta)} + s_x^{(c)} = s_b^{(c)}$ are determined by

$$S_\infty^c(G) = \begin{cases} cl^c(b), \text{ if } \dfrac{\gamma}{b\beta} \in c_0, \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

ii) Let $F = c$, or ℓ_∞. The solutions of the (SSE) $\left(s_a^0\right)_{G(\alpha,\beta)} + F_x = F_b$ and $\left(\ell_a^p\right)_{G(\alpha,\beta)} + F_x = F_b$ are determined by $S_0^F(G)$ and $S_p^F(G)$. We have $S_0^F(G) = S_p^F(G)$ with $(p \geq 1)$ and

$$S_p^F(G) = \begin{cases} cl^F(b), \text{ if } \dfrac{\gamma}{b\beta} \in \ell_\infty, \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

Corollary 3.5. *Let a, b, α, $\beta \in U^+$ and assume that either $\gamma = a/\alpha$, or $b\beta$ is a nondecreasing sequence. Then the solutions of the (SSE) $\left(W_a^0\right)_{G(\alpha,\beta)} + s_x = s_b$ are determined by*

$$S_{w_0}^{\infty}(G) = \begin{cases} cl^{\infty}(b), \text{ if } \left(n\frac{\gamma_n}{b_n\beta_n}\right)_{n\geq 1} \in \ell_\infty, \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

3.2. Application to the (SSE) involving the operators $C(\lambda)$, and the operator of the weighted means $\overline{N}_q$

In this part, we consider the operators $C(\lambda)$, or $\overline{N}_q$ that are factorable matrices determined in the following way. We have $C(\lambda) = G(1/\lambda, e)$ for $\lambda \in U^+$, and for any given positive sequence q, we let $Q_n = \sum_{k=1}^{n} q_k$ and $\overline{N}_q = G(1/Q, q)$. So the nonzero entries of the triangle $\overline{N}_q$ are given by $\left[\overline{N}_q\right]_{nk} = q_k/Q_n$ with $k \leq n$, for all n, and $\overline{N}_q$ is called the matrix of the weighted means. We use the notation

$$v_n = (\lambda_n a_n + \lambda_{n-1}a_{n-1})/b_n.$$

As a direct consequence of Theorem 3.1, we obtain the following results.

Corollary 3.6. *Let a, b, $\lambda \in U^+$. Then we have:*

i) The solutions of the (SSE) $(s_a)_{C(\lambda)} + s_x^{(c)} = s_b^{(c)}$ are determined by

$$S_\infty^c(C(\lambda)) = \begin{cases} cl^c(b), \text{ if } \lim_{n\to\infty} v_n = 0, \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

ii) Let $F = c$, or ℓ_∞. The solutions of the (SSE) $\left(s_a^0\right)_{C(\lambda)} + F_x = F_b$ and $\left(\ell_a^p\right)_{C(\lambda)} + F_x = F_b$ with $p \geq 1$, are determined by $S_0^F(C(\lambda))$ and $S_p^F(C(\lambda))$. We have $S_p^F(C(\lambda)) = S_0^F(C(\lambda))$ and

$$S_p^F(C(\lambda)) = \begin{cases} cl^F(b), \text{ if } \sup_{n\geq 1} v_n < \infty, \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

iii) The solutions of the (SSE) defined by $\left(W_a^0\right)_{C(\lambda)} + s_x = s_b$ are determined by

$$S_{w_0}^{\infty}(C(\lambda)) = \begin{cases} cl^{\infty}(b), \text{ if } \sup_{n\geq 1}(nv_n) < \infty, \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

Remark 3.7. *For $a, b, q \in U^+$, the sets $S_\infty^c\left(\overline{N}_q\right)$ and $S_p^F\left(\overline{N}_q\right) = S_0^F\left(\overline{N}_q\right)$ for $F = c$, or ℓ_∞ can be obtained as above replacing v_n by*

$$\rho_n = (a_n Q_n + a_{n-1}Q_{n-1})/q_n b_n.$$

Under some additional hypotheses and using Corollary 3.5, we obtain more simple expressions for $S_{w_0}^{\infty}(C(\lambda))$ and $S_{w_0}^{\infty}\left(\overline{N}_q\right)$ that are stated as follows.

Corollary 3.8. *Let a, b, $\lambda, q \in U^+$. Then we have:*

i) If either $a\lambda$, or b is a nondecreasing sequence, then the set $S_{w_0}^{\infty}(C(\lambda))$ of all the solutions of the (SSE) $\left(W_a^0\right)_{C(\lambda)} + s_x = s_b$ is determined by

$$S_{w_0}^{\infty}(C(\lambda)) = \begin{cases} cl^{\infty}(b), \text{ if } \left(n\lambda_n\frac{a_n}{b_n}\right)_{n\geq 1} \in \ell_\infty, \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

ii) If either a, or bq is a nondecreasing sequence, then the set $S_{w_0}^{\infty}\left(\overline{N}_q\right)$ *of all the solutions of the (SSE)* $\left(W_a^0\right)_{\overline{N}_q}+s_x = s_b$ *is determined by*

$$S_{w_0}^{\infty}\left(\overline{N}_q\right) = \begin{cases} cl^{\infty}(b), \text{ if } \left(n\dfrac{Q_n}{q_n}\dfrac{a_n}{b_n}\right)_{n\geq 1} \in \ell_\infty, \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

As a direct consequence of Corollary 3.6 for $\lambda = e$, we obtain the next result.

Corollary 3.9. *Let* $a, b \in U^+$.

i) The (SSE) $(s_a)_\Sigma + s_x^{(c)} = s_b^{(c)}$ *has solutions if and only if* $\lim_{n\to\infty} a_{n-i}/b_n = 0$ *for* $i = 0$, *or* 1, *that are determined by* $S_\infty^c(\Sigma) = cl^c(b)$.

ii) Let $F = c$, *or* ℓ_∞. *Then the sets* $S_p^F(\Sigma)$ *and* $S_0^F(\Sigma)$ *of all the solutions of the (SSE)* $\left(\ell_a^p\right)_\Sigma + F_x = F_b$ *and* $\left(s_a^0\right)_\Sigma + F_x = F_b$, *satisfy* $S_p^F(\Sigma) = S_0^F(\Sigma)$ *for* $p \geq 1$. *Then the (SSE)* $\left(s_a^0\right)_\Sigma + F_x = F_b$ *has solutions if and only if* $\sup_n (a_{n-i}/b_n) < \infty$ *for* $i = 0$, *or* 1, *that are determined by* $S_0^F(\Sigma) = cl^F(b)$.

iii) The (SSE) $\left(W_a^0\right)_\Sigma + s_x = s_b$ *has solutions if and only if* $\sup_n (na_{n-i}/b_n) < \infty$ *for* $i = 0$, *or* 1, *that are determined by* $S_{w_0}^{\infty}(\Sigma) = cl^{\infty}(b)$.

As a direct consequence of Corollary 3.8, we obtain the next result.

Remark 3.10. *Let* $b, q \in U^+$ *and let* $\mathcal{A}_b^N$ *be the set of all positive nondecreasing sequences a such that the equation* $s_x = s_b$ *and the perturbed equation* $\left(W_a^0\right)_{\overline{N}_q} + s_x = s_b$ *have the same solutions. By Corollary 3.8, we obtain*

$$\mathcal{A}_b^N = s^+_{\left(\frac{b_n q_n}{nQ_n}\right)_{n\geq 1}}.$$

Using Corollary 3.8 for $a = e$, we obtain the next result.

Corollary 3.11. *Let* b, λ, $q \in U^+$. *Then we have:*

i) If λ *is a nondecreasing sequence, then the (SSE)* $s_x = s_b$ *and the perturbed equation* $(w_0)_{C(\lambda)} + s_x = s_b$ *have the same solutions if and only if* $\left(n\lambda_n b_n^{-1}\right)_{n\geq 1} \in \ell_\infty$.

ii) The (SSE) $s_x = s_b$ *and the perturbed equation* $(w_0)_{\overline{N}_q} + s_x = s_b$ *have the same solutions if and only if* $\left(nQ_n b_n^{-1} q_n^{-1}\right)_{n\geq 1} \in \ell_\infty$.

Remark 3.12. *Let* $b \in U^+$ *and let a be a nondecreasing sequence. From Corollary 3.9 iii) and Corollary 3.8 with* $\lambda = e$, *we can easily see that the (SSE)* $s_x = s_b$ *and the perturbed equation* $\left(W_a^0\right)_\Sigma + s_x = s_b$ *have the same solutions if and only if* $(na_n/b_n)_{n\geq 1} \in \ell_\infty$. *For* $a, b \in U^+$ *where b is a nondecreasing sequence the previous result remains true. In this way, for* $b = e$, *we may notice that the solutions of the (SSE)* $\left(W_a^0\right)_\Sigma + s_x = \ell_\infty$ *and* $s_x = \ell_\infty$ *are equivalent if and only if* $a \in s_{(1/n)_{n\geq 1}}$.

Example 3.13. *Let* $r, u > 0$. *Using Corollary 3.8 (i), the set* $S_{w_0}^{\infty}(C_1)$ *of all the solutions of the (SSE)* $\left(W_r^0\right)_{C_1} + s_x = s_u$ *is determined by*

$$S_{w_0}^{\infty}(C_1) = \begin{cases} cl^{\infty}(u), \text{ if } r < u, \\ \varnothing, \quad \text{if } r \geq u. \end{cases}$$

Example 3.14. *Let* $\xi > 0$. *Using Corollary 3.8 (i) and the fact that*

$$\left(n\lambda_n \frac{a_n}{b_n}\right)_{n\geq 1} = \left(\frac{n^2}{n^{\xi}}\right)_{n\geq 1} \in \ell_\infty \quad \text{if and only if} \quad \xi \geq 2,$$

the (SSE) $(w_0)_{C_1} + s_x = s_{\left(n^{\xi}\right)_{n\geq 1}}$ *has solutions if and only if* $\xi \geq 2$.

4. On the (SSE) involving the operator Σ^2

In this section, we solve the (SSE) with operator of the form $(E_a)_{\Sigma^2}+F_x = F_b$ where $a, b \in U^+$ and $(E, F) \in \Theta$. We give an extension of i) and ii) in Corollary 3.9 and we solve the (SSE) defined by $(s_a)_{\Sigma^2} + s_x^{(c)} = s_b^{(c)}$, $\left(\ell_a^p\right)_{\Sigma^2} + F_x = F_b$ and $\left(s_a^0\right)_{\Sigma^2} + F_x = F_b$, with $F = c$, or ℓ_∞. It can easily be seen that the nonzero entries of the triangle Σ^2 are defined by $\left(\Sigma^2\right)_{nk} = n - k + 1$ for $k \leq n$ for all n. For instance the (SSE) $\left(s_a^0\right)_{\Sigma^2} + s_x^{(c)} = s_b^{(c)}$ is equivalent to the next statement. For every $y \in \omega$, we have $y_n/b_n \to l$ if and only if there are $u, v \in \omega$ with $y = u + v$ such that $\left(\sum_{k=1}^{n} k u_{n-k+1}\right)/a_n \to 0$ and $v_n/x_n \to l'$ $(n \to \infty)$ for some scalars l and l'. As a direct consequence of Theorem 2.3, we obtain the next result.

Corollary 4.1. *Let $a, b \in U^+$. Then we have:*

i) The solutions of the (SSE) $(s_a)_{\Sigma^2} + s_x^{(c)} = s_b^{(c)}$ are determined by

$$S_\infty^c\left(\Sigma^2\right) = \begin{cases} cl^c(b), \text{ if } (a_{n-i}/b_n)_{n\geq1} \in c_0 \text{ for } i = 0,1,2, \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

ii) Let $F = c$, or ℓ_∞. Then the sets $S_p^F\left(\Sigma^2\right)$ and $S_0^F\left(\Sigma^2\right)$ of all the solutions of the (SSE) $\left(\ell_a^p\right)_{\Sigma^2} + F_x = F_b$ and $\left(s_a^0\right)_{\Sigma^2} + F_x = F_b$, satisfy $S_p^F\left(\Sigma^2\right) = S_0^F\left(\Sigma^2\right)$ for $p \geq 1$ and

$$S_p^F\left(\Sigma^2\right) = \begin{cases} cl^F(b), \text{ if } (a_{n-i}/b_n)_{n\geq1} \in \ell_\infty \text{ for } i = 0,1,2, \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

Proof. First it can easily be seen that the inverse of Σ^2 is the trangle Δ^2 whose the nonzero entries are determined by $\left(\Delta^2\right)_{nn} = 1$ for all n, $\left(\Delta^2\right)_{n,n-1} = -2$ for $n \geq 2$, and $\left(\Delta^2\right)_{n,n-2} = 1$ for $n \geq 3$. Then Δ^2 satisfies the condition in (4). So the proofs of i) and ii) are direct consequences of Theorem 2.3. Since a, b are positive sequences, the proof of i) follows from the equivalence of the conditions $\lim_{n\to\infty} (a_{n-2} + 2a_{n-1} + a_n)/b_n = 0$ and $\lim_{n\to\infty} (a_{n-i}/b_n) = 0$ for $i = 0, 1, 2$, and the proof of ii) follows from the equivalence of each of the conditions

$$\sup_{n\geq1}\left\{(a_{n-2} + 2a_{n-1} + a_n)\, b_n^{-1}\right\} < \infty, \quad \sup_{n\geq1}\left\{\left(a_{n-2}^q + 2^q a_{n-1}^q + a_n^q\right) b_n^{-q}\right\} < \infty$$

and $\sup_{n\geq1} (a_{n-i}/b_n) < \infty$ for $i = 0, 1, 2$. □

Example 4.2. *As a direct consequence of Corollary 4.1 ii), the set* S *of all positive sequences a for which we have the equivalence of the perturbed equation $\left(s_a^0\right)_{\Sigma^2} + s_x^{(c)} = s_b^{(c)}$ and the equation $s_x^{(c)} = s_b^{(c)}$ is determined by*

$$\mathrm{S} = s_b^+ \cap s^+_{(b_{n+1})_{n\geq1}} \cap s^+_{(b_{n+2})_{n\geq1}}.$$

Example 4.3. *As a direct consequence of Corollary 4.1 ii), it can easily be shown that the (SSE) defined by*

$$\left(\ell^p_{(1/n!)_{n\geq1}}\right)_{\Sigma^2} + s_x^{(c)} = s^{(c)}_{\left(n^h (n!)^{-1}\right)_{n\geq1}}$$

for reals p and h with $p > 1$ and $h > 0$ has solutions if and only if $h \geq 2$.

Remark 4.4. *As a direct consequence of Corollary 4.1 ii) for $a = e$, the (SSE) defined by $(c_0)_{\Sigma^2} + s_x = s_b$ and $\ell^p_{\Sigma^2} + s_x = s_b$, $(p \geq 1)$ are equivalent. More precisely, each of these (SSE) has solutions if and only if $1/b \in l_\infty$. As a direct consequence of Corollary 4.1 i) for $a = e$, the (SSE) $(\ell_\infty)_{\Sigma^2} + s_x^{(c)} = s_b^{(c)}$ has solutions if and only if $b_n \to \infty$ $(n \to \infty)$.*

References

[1] Başar F., Malkowsky, E., Bilâl A., *Matrix transformations on the matrix domains of triangles in the spaces of strongly C_1 summable and bounded sequences* Publicationes Math. **78** (2008),193-213.
[2] de Malafosse, B., *On some BK space*, Int. J. of Math. and Math. Sci. **28** (2003), 1783-1801.
[3] de Malafosse, B., *Sum of sequence spaces and matrix transformations,* Acta Math. Hung. **113** (**3**) (2006), 289-313.
[4] de Malafosse, B., *Application of the infinite matrix theory to the solvability of certain sequence spaces equations with operators.* Mat. Vesnik, **54** (**1**) (2012), 39-52.
[5] de Malafosse, B., *Applications of the summability theory to the solvability of certain sequence spaces equations with operators of the form $B(r,s)$*. Commun. Math. Anal. **13** (**1**) (2012), 35-53.
[6] de Malafosse B., *Solvability of certain sequence spaces inclusion equations with operators,* Demonstratio Math. **46** (**2**) (2013), 299-314.
[7] de Malafosse, B., *Solvability of sequence spaces equations using entire and analytic sequences and applications,* J. Indian Math. Soc. **81** (**1-2**) (2014), 97-114.
[8] de Malafosse, B., *On sequence spaces equations of the form $E_T + F_x = F_b$ for some triangle T*, Jordan J. Math. Stat. **8** (**1**) (2015) 79-105.
[9] de Malafosse, B., *On the spectra of the operator of the first difference on the spaces W_τ and W_τ^0 and application to matrix transformations,* Gen. Math. Notes, **22** (**2**) (2014) 7-21.
[10] de Malafosse, B., *Some general results on the solvability of sequence spaces equations with applications*, Submitted in Malaysian J. Math.
[11] de Malafosse, B., *Solvability of sequence spaces equations of the form $(E_a)_\Delta + F_x = F_b$*, Fasc. Math. **55** (2015), 109-131.
[12] de Malafosse, B., *New results on the sequence spaces equations using the operator of the first difference,* Acta math. Univ. Comen. **86** (**2**) (2017) 227-242.
[13] de Malafosse, B., *Application of the infinite matrix theory to the solvability of sequence spaces inclusion equations with operators,* Ukr. Mat. Zh. **71** (**8**) (2019), 1040–1052.
[14] de Malafosse, B., Malkowsky, E., *Matrix transformations between sets of the form W_ξ and operators generators of analytic semigroups,* Jordan J. math. stat. **1** (**1**) (2008) 51-67.
[15] de Malafosse, B., Malkowsky, E., *On the solvability of certain (SSIE) with operators of the form $B(r,s)$,* Math. J. Okayama. Univ. **56** (2014), 179-198.
[16] de Malafosse, B., Malkowsky, E., *On sequence spaces equations using spaces of strongly bounded and summable sequences by the Cesàro method* Antartica J. Math. **10** (**6**) (2013), 589-609.
[17] de Malafosse, B., Rakočević V., *Calculations in new sequence spaces and application to statistical convergence*, Cubo A, **12** (**3**) (2010), 117- 132.
[18] de Malafosse, B., Rakočević V., *Matrix transformations and sequence spaces equations.* Banach J. Math. Anal. **7** (**2**) (2013), 1-14.
[19] de Malafosse, B., Fares, A., Ayad, A., *Matrix transformations and application to perturbed problems of some sequence spaces equations with operators*, Filomat, **32** (**14**) (2018), 5123-5130.
[20] Farés A., de Malafosse, B., *Sequence spaces equations and application to matrix transformations* Int. Math. Forum, **3** (**19**) (2008), 911-927.
[21] Malkowsky, E., *The continuous duals of the spaces $c_0(\Lambda)$ and $c(\Lambda)$ for exponentially bounded sequences Λ* , Acta Sci. Math. (Szeged), **61** (1995), 241-250.
[22] Malkowsky, E., Rakočević, V., *An introduction into the theory of sequence spaces and measure of noncompactness*, Zbornik radova, Matematički institut SANU, **9** (**17**) (2000), 143-243.
[23] Maddox, I.J., *Infinite matrices of operators,* Springer-Verlag, Berlin, Heidelberg and New York, 1980.
[24] Wilansky, A., *Summability through Functional Analysis,* North-Holland Mathematics Studies 85, 1984.