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Solvability of infinite differential systems of the form $x'(t) = Tx(t) + b$ where T is either of the triangles $C(\lambda)$ or $\overline{N_q}$

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Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

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Abstract

In this paper, we are interested in solving infinite linear systems of differential equations of the form $x'(t) = Tx(t) + b$ with $x(0) = x_0$, where T is either the generalized Cesàro operator $C(\lambda)$ or the weighted mean matrix $\overline{N_q}$, x_0 and b are two given infinite column matrices and λ is a sequence with non-zero entries. We use a new method based on Laplace transformations to solve these systems.

Keywords: Infinite linear systems of differential equations, systems of linear equations, Laplace operator.
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1. Introduction

In our previous paper [4], we dealt with solving infinite linear differential systems of the form $x'(t) = \Delta(\lambda)x(t) + b$ with $x(0) = x_0$, where $\Delta(\lambda)$ is the triangle defined by:

$$\Delta(\lambda)_{nk} = \begin{cases} \lambda_n & \text{for } k = n, \\ -\lambda_{n-1} & \text{for } k = n-1, \\ 0 & \text{for } k \neq n-1 \text{ and } k \neq n \ (n \geq 1), \end{cases}$$

with $\lambda_0 = 0$, x_0 and b are two given infinite column matrices, $\lambda = (\lambda_n)_{n \geq 1}$ is a sequence with non-zero terms and $x(t) = (x_n(t))_{n \geq 1}$ is the unknown sequence of functions. We have considered two cases for study, in

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the first one, we suppose that λ is a constant sequence and in the second case, all terms of λ are supposed to be pairwisely distinct. Solutions of such systems are explicitly given using Laplace transformation.

In this paper, we deal with infinite linear systems of differential equations of the form:

$$x'(t) = C(\lambda)x(t) + b, \tag{1.1}$$

and

$$x'(t) = \overline{N}_q \, x(t) + b, \tag{1.2}$$

where $C(\lambda) = \Delta(\lambda)^{-1}$ is the generalized Cesàro operator and $\overline{N}_q$ is the weighted mean matrix. Recall that for the operator of weighted mean, $q = (q_n)_n$ is a sequence of positive entries, and $Q = (\sum_{m=1}^{n} q_m)_n$. Then $\overline{N}_q = D_{1/Q}\Sigma D_q$ where $\Sigma = C(e) = \Delta(e)^{-1}$ for $e = (1, \ldots, 1, \ldots)$ and D_q (resp. $D_{1/Q}$) is the infinite diagonal matrix where the entries on the main diagonal are the terms of the sequence q (resp. $1/Q$).

The Cesàro operator was studied by Hausdorff, Leibowitz, Reade [10], Okutoyi [8] and de Malafosse [3]. In these papers the authors gave results on the spectrum of this matrix. Note that in [5] can be found other results on Cesàro operator. Infinite matrix theory is used in many branches of classical mathematics such as infinite quadratic forms, integral equations, matrix transformations, differential equations, operators between sequence spaces, it is also used to provide approximations of solutions. Infinite-dimensional linear systems appear naturally when studying control problems for systems modelled by linear partial differential equations. Many problems in dynamic systems can be written in the form of differential equations or infinite differential systems and lead to infinite linear systems. In this way, we cite Hill's equation that was studied by L. Brillouin, E. L. Ince [6], K.G. Valeev [12], H. Hochstadt (1963), S. Winkler (1966) and B. Rossetto [11]. This equation is the second order differential equation of the form

$$y''(z) + J(z)y(z) = 0, \tag{1.3}$$

where $z \in \Omega$ and Ω is an open subset of $\mathbb{C}$, containing the real axis and $J(z) = \sum_{n=-\infty}^{+\infty} \theta_n e^{2inz}$ is a special periodic function, where $\theta_n = \theta_{-n}$ for all $n \in \mathbb{Z}$. It was shown that the solutions of (1.3) are of the form $y(z) = e^{\mu z} \sum_{m=-\infty}^{+\infty} x_m e^{2imz}$ where $\mu \in \mathbb{C}$ is the Floquet exponent. Replacing $y(z)$ by its expression in equation (1.3), we obtain an infinite linear system represented by the matrix equation

$$A_\mu x = 0, \tag{1.4}$$

where $x^t = (\ldots, x_{-1}, x_0, x_1, \ldots)$ and $A_\mu = (a_{nm})_{n,m \in \mathbb{Z}}$ is an infinite matrix depending on μ (cf. [1]) defined by:

$$\begin{cases} a_{nn} = \theta_0 + (\mu + 2ni)^2, & \forall n \in \mathbb{Z} \\ a_{nm} = \theta_{|m-n|}, & \text{for } m \neq n. \end{cases}$$

The aim is then to determine the values of μ for which (1.4) has a non trivial solution. Some authors determined such values of μ using an *infinite determinant* [9]. B. Rossetto provided an algorithm that allows to calculate the Floquet exponent from the generalization of the notion of the *characteristic equation and of a truncated determinant*. On the other hand, B. de Malafosse [2] dealt with equation (1.3) and studied system (1.4) using special additionnal equations. More recently, B. de Malafosse [1] used the same method for the study of the Mathieu equation.

This paper is organized as follows. In Section 2 we define the triangle matrix and we recall some results on infinite bidiagonal matrices and Laplace tranformation operator. Section 3 is devoted to the resolution of system (1.1). In Subsection 3.1 we consider the case when λ is constant, then in Subsection 3.2 we deal with the case when the λ_i are all distincts. In Section 4, we will apply these results to the resolution of equation (1.2).

2. Preliminaries

In this paper, we consider infinite lower triangular matrices with nonzero diagonal entries that are called triangles. An infinite matrix $T = (t_{nk})_{n,k \geq 1}$ is a triangle if and only if $t_{nk} = 0$ for all $k > n$ and $t_{nn} \neq 0$ for all $n \geq 1$.

We denote by w the set of all the sequences and by U the set of the sequences $u = (u_n)_{n \geq 1}$ with $u_n \neq 0$ for all $n \geq 1$. The matrix T is considered as an operator from w to itself in the following way, for every sequence $x \in w$ which can be written as a column matrix $x = (x_1, \ldots, x_n, \ldots)^t$, we have $Tx = (T_1(x), \ldots, T_n(x), \ldots)^t$ with

$$T_n(x) = \sum_{k=1}^{n} t_{nk} x_k \quad \text{for all } n \geq 1.$$

It is known that every triangle T is invertible and if T^{-1} denotes its inverse then we have

$$T(T^{-1}x) = T^{-1}(Tx) = x \quad \text{for all } x \in w. \tag{2.1}$$

We are interested in solving infinite linear systems represented by

$$Tx = b \tag{2.2}$$

for a given $b \in w$ where $x \in w$ is the unknown. Equation (2.2) is equivalent to

$$\sum_{k=1}^{n} t_{nk} x_k = b_n \quad n = 1, 2, \ldots$$

It can be easily deduced from (2.1) that the unique solution of (2.2) is given by

$$x = T^{-1}b.$$

In this paper, we will solve the equation $\Delta(\alpha, \beta)x = b$ where $\Delta(\alpha, \beta)$ is the triangle defined by

$$\Delta(\alpha, \beta) = \begin{pmatrix} \alpha_1 & & & & \mathbf{0} \\ -\beta_1 & \alpha_2 & & & \\ & \ddots & \ddots & & \\ & & -\beta_{n-1} & \alpha_n & \\ \mathbf{0} & & & \ddots & \ddots \end{pmatrix}$$

and $\alpha = (\alpha_n)_{n \geq 1} \in U$ and $\beta = (\beta_n)_{n \geq 1} \in w$.

Lemma 2.1. *Let $\alpha = (\alpha_n)_{n \geq 1} \in U$ and $\beta = (\beta_n)_{n \geq 1} \in \omega$. We have*

$$(\Delta(\alpha, \beta))^{-1} = C(\alpha, \beta) = (c_{nk})_{n,k \geq 1},$$

where

$$c_{nk} = \begin{cases} \dfrac{1}{\alpha_n} & \text{if } k = n, \\[2ex] \dfrac{1}{\alpha_n} \displaystyle\prod_{i=k}^{n-1} \dfrac{\beta_i}{\alpha_i} & \text{if } k < n, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Let $\Delta(\alpha) = \Delta(\alpha, \alpha)$ and $C(\alpha) = C(\alpha, \alpha)$. Then $\Delta(\alpha)$ is the triangle defined by:

$$\Delta(\alpha)_{nk} = \begin{cases} \alpha_n & \text{for } k = n, \\ -\alpha_{n-1} & \text{for } k = n-1, \\ 0 & \text{for } k \neq n-1 \text{ and } k \neq n \ (n \geq 1), \end{cases}$$

and $C(\alpha)$ is the triangle defined by:

$$C(\alpha)_{n,k} = \begin{cases} 1/\alpha_n & \text{for } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $C(\alpha)$ is the inverse of $\Delta(\alpha)$.

Let $e \in U$, defined by $e_n = 1$ for all $n \geq 1$. Then $\Delta = \Delta(e)$ is the well known operator of the first-difference defined by

$$\Delta_n(x) = x_n - x_{n-1} \text{ for all } n \geq 1,$$

with the convention $x_0 = 0$. Recall that the operator Δ is invertible and its inverse is usually written $\Sigma = C(e)$.

Finally, we recall some properties on Laplace transformations that are useful in the sequel. For a function f of one variable t, we define the Laplace transformation of f as follows:

$$F(p) = \int_0^{+\infty} f(t)\, e^{-pt} dt,$$

where $p \in \mathbb{C}$ is a new variable. We denote by $\pounds : f \longmapsto F$, the Laplace operator.

Lemma 2.2. *Let $\pounds^{-1}$ be the inverse mapping of $\pounds$. Let $c \in \mathbb{R}$ and f be a function of one variable t. Then the following properties hold:*

1. *$\pounds$ and $\pounds^{-1}$ are linear operators.*
2. *$\pounds(c) = c/p$.*
3. *$\pounds(f'(t)) = p\,\pounds(f(t)) - f(0)$.*
4. *For all $m \in \mathbb{N}$,*

$$\pounds^{-1}\left(\frac{1}{(p-c)^m}\right) = \frac{t^{m-1}e^{ct}}{(m-1)!}.$$

3. The equation $x'(t) = C(\lambda)\,x(t) + b$

Let $\lambda = (\lambda_n)_{n \geq 1} \in U$ be a sequence. In this section, we are interested in the study of the equation:

$$\begin{cases} x'(t) & = & C(\lambda)\,x(t) + b, \\ x(0) & = & x_0, \end{cases} \tag{3.1}$$

where $x_0 = (a_n)_{n \geq 1}$ and $b = (b_n)_{n \geq 1}$ are two given sequences.
In the sequel, we will divide our study into two cases:

- Case where $\lambda_n = c \neq 0$ for all $n \geq 1$.

- Case where λ_n ($n \geq 1$) are pairwisely distinct.

3.1. The equation $x'(t) = \frac{1}{c}\Sigma x(t) + b$

If $\lambda_n = c \neq 0$ for all $n \geq 1$ then $C(\lambda) = \frac{1}{c}\Sigma$, where $\Sigma = C(e)$, $e_n = 1$ for all $n \geq 1$ and $\Sigma^{-1} = \Delta$ is the operator of the first-difference.

The main result of this subsection is stated in the following theorem:

Theorem 3.1. *Infinite differential system (3.1) has a unique solution which is given for each $n \geq 1$, by*

$$x_n(t) = e^{t/c} \sum_{k=1}^{n} \Delta_k(x_0) \sum_{s=0}^{n-k} C_{n-k}^s \frac{1}{c^s s!} t^s$$

$$+ e^{t/c} \sum_{k=1}^{n} \Delta_k(b) \sum_{s=0}^{n-k-1} C_{n-k-1}^s \frac{1}{c^s(s+1)!} t^{s+1}.$$

where $\Delta_k(x_0) = a_k - a_{k-1}$ and $\Delta_k(b) = b_k - b_{k-1}$ for all $k \geq 1$ with the convention that $a_0 = b_0 = 0$.

Proof. Multiply equation (3.1) by c, and apply Δ to each member of the first equation, we obtain

$$\begin{cases} c\,\Delta x'(t) = x(t) + c\,\Delta b \\ x(0) = x_0, \end{cases}$$

that is the following infinite linear system:

$$\begin{cases} c\,x'_n(t) - c\,x'_{n-1}(t) = x_n(t) + c\,\Delta_n(b) \\ x_n(0) = a_n, \quad n = 1, 2, \dots \end{cases} \tag{3.2}$$

where $x'_0(t) = 0$, $b_0 = 0$ and $\Delta_n(b) = b_n - b_{n-1}$ for all $n \geq 1$.
By applying Laplace operator to equations (3.2), we get:

$$(c\,p - 1)X_n - c\,pX_{n-1} = c\left(\Delta_n(x_0) + \frac{\Delta_n(b)}{p}\right), \quad n = 1, 2, \dots \tag{3.3}$$

where $X_n = X_n(p) = \mathcal{L}(x_n(t))$, $\Delta_n(x_0) = a_n - a_{n-1}$, with the convention that $a_0 = 0$ and $X_0 = 0$.
Equations (3.3) are equivalent to the following linear system:

$$\Delta(c\,p - 1, c\,p)\,X = B$$

where $X^t = (X_n)_{n \geq 1}$, $B^t = (B_n)_{n \geq 1}$ and

$$B_n = c\left(\Delta_n(x_0) + \frac{\Delta_n(b)}{p}\right), \quad n \geq 1$$

and $\Delta(c\,p - 1, c\,p)$ is the triangle

$$\Delta(c\,p - 1, c\,p) = \begin{pmatrix} c\,p - 1 & & & \\ -c\,p & c\,p - 1 & & 0 \\ & & \cdot & \cdot \\ 0 & & -c\,p & c\,p - 1 \\ & & & \cdot & \cdot \end{pmatrix}.$$

So by Lemma 2.1 we obtain:

$$\Delta(c\,p - 1, c\,p)^{-1} = \begin{pmatrix} \frac{1}{c\,p-1} & & & \\ \frac{c\,p}{(c\,p-1)^2} & \frac{1}{c\,p-1} & & 0 \\ \cdot & \cdot & \cdot & \\ \frac{(c\,p)^{n-1}}{(c\,p-1)^n} & \cdot & \frac{c\,p}{(c\,p-1)^2} & \frac{1}{c\,p-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

This means that

$$\left[\Delta\left(c\,p-1,c\,p\right)^{-1}\right]_{n,k} = \frac{(c\,p)^{n-k}}{(c\,p-1)^{n-k+1}} \quad \text{for } k \leq n;$$

using Newton's formula we have

$$(c\,p)^{n-k} = (c\,p-1+1)^{n-k} = \sum_{i=0}^{n-k} C_{n-k}^i \,(c\,p-1)^i,$$

then we have

$$
\begin{aligned}
X_n &= \sum_{k=1}^{n} \frac{(c\,p)^{n-k}}{(c\,p-1)^{n-k+1}} B_k \\
&= \sum_{k=1}^{n} c\left(\Delta_k(x_0) + \frac{\Delta_k(b)}{p}\right)\frac{(c\,p)^{n-k}}{(c\,p-1)^{n-k+1}} \\
&= \sum_{k=1}^{n} c\,\Delta_k(x_0)\frac{(c\,p)^{n-k}}{(c\,p-1)^{n-k+1}} + \sum_{k=1}^{n} c\left(\frac{\Delta_k(b)}{p}\right)\frac{(c\,p)^{n-k}}{(c\,p-1)^{n-k+1}} \\
&= \sum_{k=1}^{n} c\,\Delta_k(x_0)\frac{(c\,p)^{n-k}}{(c\,p-1)^{n-k+1}} + \sum_{k=1}^{n} c^2\Delta_k(b)\frac{(c\,p)^{n-k-1}}{(c\,p-1)^{n-k+1}} \\
&= \sum_{k=1}^{n} c\,\Delta_k(x_0)\frac{\sum_{i=0}^{n-k} C_{n-k}^i\,(c\,p-1)^i}{(c\,p-1)^{n-k+1}} + \sum_{k=1}^{n} c^2\Delta_k(b)\frac{\sum_{i=0}^{n-k-1} C_{n-k-1}^i\,(c\,p-1)^i}{(c\,p-1)^{n-k+1}} \\
&= \sum_{k=1}^{n} c\,\Delta_k(x_0)\sum_{i=0}^{n-k} C_{n-k}^i\frac{1}{(c\,p-1)^{n-k-i+1}} \\
&\quad + \sum_{k=1}^{n} c^2\Delta_k(b)\sum_{i=0}^{n-k-1} C_{n-k-1}^i\frac{1}{(c\,p-1)^{n-k-i+1}} \\
&= \sum_{k=1}^{n} c\,\Delta_k(x_0)\sum_{i=0}^{n-k} C_{n-k}^i\frac{1}{\left[c\left(p-\frac{1}{c}\right)\right]^{n-k-i+1}} \\
&\quad + \sum_{k=1}^{n} c^2\Delta_k(b)\sum_{i=0}^{n-k-1} C_{n-k-1}^i\frac{1}{\left[c\left(p-\frac{1}{c}\right)\right]^{n-k-i+1}}.
\end{aligned}
$$

Since we have

$$\mathcal{L}^{-1}\left[\frac{1}{\left[c\left(p-\frac{1}{c}\right)\right]^{n-k-i+1}}\right] = \frac{1}{c^{n-k-i+1}}\frac{t^{n-k-i}}{(n-k-i)!}e^{t/c},$$

then

$$
\begin{aligned}
x_n(t) &= \mathcal{L}^{-1}[X_n] \\
&= \mathcal{L}^{-1}\left[\sum_{k=1}^{n} c\,\Delta_k(x_0)\sum_{i=0}^{n-k} C_{n-k}^{i}\frac{1}{\left[c\left(p-\frac{1}{c}\right)\right]^{n-k-i+1}}\right] \\
&\quad +\mathcal{L}^{-1}\left[\sum_{k=1}^{n} c^2\,\Delta_k(b)\sum_{i=0}^{n-k-1} C_{n-k-1}^{i}\frac{1}{\left[c\left(p-\frac{1}{c}\right)\right]^{n-k-i+1}}\right] \\
&= e^{t/c}\sum_{k=1}^{n}\Delta_k(x_0)\sum_{i=0}^{n-k} C_{n-k}^{i}\frac{1}{c^{n-k-i}}\frac{t^{n-k-i}}{(n-k-i)!} \\
&\quad +c\,e^{t/c}\sum_{k=1}^{n}\Delta_k(b)\sum_{i=0}^{n-k-1} C_{n-k-1}^{i}\frac{1}{c^{n-k-i}}\frac{t^{n-k-i}}{(n-k-i)!}.
\end{aligned}
$$

Now let $s = n-k-i$. Since $C_{n-k}^{n-k-s} = C_{n-k}^{s}$, we obtain

$$
\begin{aligned}
x_n(t) &= e^{t/c}\sum_{k=1}^{n}\Delta_k(x_0)\sum_{s=0}^{n-k} C_{n-k}^{s}\;\frac{1}{c^s\,s!}\;t^s \\
&\quad +c\,e^{t/c}\sum_{k=1}^{n}\Delta_k(b)\sum_{s=0}^{n-k-1} C_{n-k-1}^{s}\;\frac{1}{c^{s+1}(s+1)!}\;t^{s+1}.
\end{aligned}
$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2. The equation $x'(t) = C(\lambda)x(t)+b$ where all λ_n are pairwisely distinct

Equation (3.1) is equivalent to the following infinite linear system:

$$
\begin{cases}
\lambda_n\,x_n'(t) &= \sum_{k=1}^{n} x_k(t)+b_n, \\
x_n(0) &= a_n, \quad n = 1, 2, \dots
\end{cases}
\tag{3.4}
$$

In the sequel, we need the following lemma:

Lemma 3.2. *Let $k, n \in \mathbb{N}$ such that $k < n$. Let $\alpha_k, \dots, \alpha_n \in \mathbb{R}$ pairwisely distinct real numbers. Then the decomposition of the function*

$$
F(z) = \frac{1}{z-\alpha_n}\prod_{j=k}^{n-1}\frac{z}{z-\alpha_j}
$$

into simple fractions is given by:

$$
F(z) = \sum_{j=k}^{n}\frac{A_j}{z-\alpha_j},
$$

where for all $k \le j \le n$,

$$
A_j = \alpha_j^{n-k}\prod_{i=k,i\neq j}^{n}\frac{1}{\alpha_j-\alpha_i}.
$$

Proof. We have

$$
A_n = \lim_{z\to\alpha_n}(z-\alpha_n)\,F(z) = \prod_{j=k}^{n-1}\frac{\alpha_n}{\alpha_n-\alpha_j} = \alpha_n^{n-k}\prod_{j=k}^{n-1}\frac{1}{\alpha_n-\alpha_j}
$$

and for all $k \le j \le n-1$,

$$
A_j = \lim_{z\to\alpha_j}(z-\alpha_j)\,F(z) = \frac{\alpha_j^{n-k}}{\alpha_j-\alpha_n}\prod_{i=k,i\neq j}^{n-1}\frac{1}{\alpha_j-\alpha_i} = \alpha_j^{n-k}\prod_{i=k,i\neq j}^{n}\frac{1}{\alpha_j-\alpha_i}
$$

$\square$

Theorem 3.3. *Infinite linear system (3.4) has a unique solution which is given for each $n \geq 1$, by*

$$x_n(t) = \frac{1}{\lambda_n}\left[B_n e^{\frac{t}{\lambda_n}} + \sum_{k=1}^{n-1} B_k \sum_{j=k}^{n} A_j e^{\frac{t}{\lambda_j}}\right] - \Delta(\lambda)_n(b), \tag{3.5}$$

where

$$B_k = \Delta(\lambda)_k\left(x_0 + \Delta(\lambda)b\right) \quad \text{for } 1 \leq k \leq n$$

and

$$A_j = \prod_{i=k, i\neq j}^{n} \frac{\lambda_i}{\lambda_i - \lambda_j} \quad \text{for } k \leq j \leq n.$$

Proof. Since $C(\lambda)^{-1} = \Delta(\lambda)$, then equation (3.1) is equivalent to the equation:

$$x'(t) = C(\lambda)\left[x(t) + \Delta(\lambda)b\right].$$

Putting $y(t) = x(t) + \Delta(\lambda)b$, then we can easily see that equation (3.1) is equivalent to the equation:

$$y'(t) = C(\lambda)y(t),$$

and then by applying $\Delta(\lambda)$, we get

$$\begin{cases} \Delta(\lambda)y'(t) - y(t) = 0, \\ y(0) = y_0, \end{cases} \tag{3.6}$$

where $y_0 = x_0 + \Delta(\lambda)b$. Then equation (3.6) is equivalent to the following infinite differential linear system:

$$\begin{cases} \lambda_n y'_n(t) - \lambda_{n-1} y'_{n-1}(t) - y_n(t) = 0 \\ y_n(0) = c_n, \quad n = 1, 2, \ldots, \end{cases} \tag{3.7}$$

where $c_n = a_n + \Delta(\lambda)_n(b)$ and $\Delta(\lambda)_n(b) = \lambda_n b_n - \lambda_{n-1} b_{n-1}$ for all $n \geq 1$, with the convention that $\lambda_0 = b_0 = 0$ and $y'_0(t) = 0$.
Applying Laplace transform to equations (3.7), we get:

$$(p\lambda_n - 1)Y_n - p\lambda_{n-1}Y_{n-1} = \Delta(\lambda)_n(y_0) \quad n = 1, 2, \ldots \tag{3.8}$$

where $Y_n = Y_n(p) = \mathcal{L}[y_n(t)]$ and $\Delta(\lambda)_n(y_0) = \lambda_n c_n - \lambda_{n-1} c_{n-1}$ for all $n \geq 1$, with the convention that $c_0 = 0$ and $Y_0 = 0$. Equations (3.8) are equivalent to the following infinite linear system:

$$\Delta(p\lambda - 1, p\lambda)Y = B,$$

where $Y^t = (Y_n)_{n\geq 1}$, $B^t = (B_n)_{n\geq 1}$ and

$$B_n = \Delta(\lambda)_n(y_0), \quad n \geq 1.$$

So by Lemma 2.1 we obtain:

$$\Delta(p\lambda - 1, p\lambda)^{-1} = \begin{pmatrix} \frac{1}{p\lambda_1 - 1} & & & & \\ \cdot & \frac{1}{p\lambda_2 - 1} & & 0 & \\ \cdot & & \cdot & \cdot & \\ \cdot & \frac{1}{p\lambda_n - 1}\prod_{j=k}^{n-1}\frac{p\lambda_j}{p\lambda_j - 1} & \cdot & \frac{1}{p\lambda_n - 1} & \\ & & & \cdot & \cdot \end{pmatrix},$$

and then $Y = \Delta(p\lambda - 1, p\lambda)^{-1}B$, thus

$$
\begin{aligned}
Y_n &= \sum_{k=1}^{n-1} \frac{1}{p\lambda_n - 1} \left(\prod_{j=k}^{n-1} \frac{p\lambda_j}{p\lambda_j - 1} \right) B_k + \frac{B_n}{p\lambda_n - 1} \\
&= \frac{1}{\lambda_n} \sum_{k=1}^{n} B_k F_k(p),
\end{aligned}
\tag{3.9}
$$

with

$$
F_k(p) = \begin{cases}
\frac{1}{p - \frac{1}{\lambda_n}} \left[\prod_{j=k}^{n-1} \frac{p}{p - \frac{1}{\lambda_j}} \right] & \text{for } 1 \le k \le n-1, \\
\\
\frac{1}{p - \frac{1}{\lambda_n}} & \text{for } k = n.
\end{cases}
$$

For $1 \le k \le n-1$, the decomposition of $F_k(p)$ into simple fractions is given by Lemma 3.2 (for $\alpha_j = 1/\lambda_j$) as follows:

$$
F_k(p) = \sum_{j=k}^{n} \frac{A_j}{p - \frac{1}{\lambda_j}},
$$

where for all $k \le j \le n$,

$$
A_j = \prod_{i=k, i \ne j}^{n} \frac{\lambda_i}{\lambda_i - \lambda_j}.
$$

Applying $\mathcal{L}^{-1}$ on (3.9), we get:

$$
y_n(t) = \mathcal{L}^{-1}(Y_n(p)) = \frac{1}{\lambda_n} \sum_{k=1}^{n} B_k \mathcal{L}^{-1}(F_k(p)).
$$

But

$$
\mathcal{L}^{-1}(F_k(p)) = \begin{cases}
\sum_{j=k}^{n} A_j e^{\frac{t}{\lambda_j}} & \text{for } 1 \le k \le n-1, \\
\\
e^{\frac{t}{\lambda_n}} & \text{for } k = n.
\end{cases}
$$

Then

$$
y_n(t) = \frac{1}{\lambda_n} \left[B_n e^{\frac{t}{\lambda_n}} + \sum_{k=1}^{n-1} B_k \sum_{j=k}^{n} A_j e^{\frac{t}{\lambda_j}} \right].
$$

Using the identity $x_n(t) = y_n(t) - \Delta(\lambda)_n(b)$, we get:

$$
x_n(t) = \frac{1}{\lambda_n} \left[B_n e^{\frac{t}{\lambda_n}} + \sum_{k=1}^{n-1} B_k \sum_{j=k}^{n} A_j e^{\frac{t}{\lambda_j}} \right] - \Delta(\lambda)_n(b),
$$

that is (3.5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Application to the equation $x'(t) = \overline{N}_q\, x(t) + b$

In this section, we will apply the previous results to the case when $T = \overline{N}_q$ is the operator of weighted mean martix. Recall that $\overline{N}_q$ is the triangle defined by

$$
[\overline{N}_q]_{nm} = \begin{cases}
\frac{q_m}{Q_n} & \text{for } m \le n, \\
0 & \text{otherwise,}
\end{cases}
$$

where $q = (q_n)_{n\geq 1}$ is a positive sequence, $Q = (Q_n)_{n\geq 1}$ is the sequence defined by $Q_n = \sum_{m=1}^{n} q_m$ for all $n \geq 1$. We can easily see that $\overline{N}_q = D_{1/Q}\Sigma D_q$ where D_q (resp. $D_{1/Q}$) is the diagonal matrix whose the n-th entry is equal to q_n (resp. the diagonal matrix whose the n-th entry is equal to $1/Q_n$). Note that the equation

$$\begin{cases} x'(t) &= \overline{N}_q\, x(t) + b, \\ x(0) &= x_0, \end{cases} \tag{4.1}$$

is equivalent to the following infinite differential system:

$$\begin{cases} x'_n(t) &= \frac{1}{Q_n}\sum_{k=1}^{n} q_k x_k(t) + b_n \\ x_n(0) &= a_n, \quad n = 1, 2, \dots \end{cases}$$

Theorem 4.1. *Equation (4.1) has a unique solution which is given for each $n \geq 1$, by*

$$x_n(t) = \frac{1}{Q_n}\left[B_n e^{\frac{q_n}{Q_n}t} + \sum_{k=1}^{n-1} B_k \sum_{j=k}^{n} A_j e^{\frac{q_j}{Q_j}t} \right] - \frac{\Delta(Q)_n(b)}{q_n}, \tag{4.2}$$

where

$$B_k = \Delta(Q)_k(x_0) + \Delta\left(\frac{Q}{q}\right)_k\left(\Delta(Q)b\right) \quad \text{for } 1 \leq k \leq n,$$

and

$$A_j = \prod_{i=k, i\neq j}^{n} \frac{Q_i}{Q_i - \frac{q_i}{q_j}Q_j} \quad \text{for } k \leq j \leq n.$$

Proof. Let $y(t) = D_q\, x(t)$, then $y'(t) = D_q\, x'(t)$. Since $\overline{N}_q = D_{1/Q}\Sigma D_q$ and $D_q D_{1/Q} = D_{q/Q}$, then equation (4.1) is equivalent to

$$\begin{cases} y'(t) &= C(\lambda)y(t) + D_q\, b, \\ y(0) &= D_q\, x_0, \end{cases} \tag{4.3}$$

taking into account that $D_{q/Q}\Sigma = C(\lambda)$, where $\lambda = Q/q$. Applying Theorem 3.3 to equation (4.3), we get:

$$y_n(t) = \frac{q_n}{Q_n}\left[B_n e^{\frac{q_n}{Q_n}t} + \sum_{k=1}^{n-1} B_k \sum_{j=k}^{n} A_j e^{\frac{q_j}{Q_j}t} \right] - \Delta(Q)_n(b),$$

where

$$\begin{aligned} B_k &= \Delta\left(\frac{Q}{q}\right)_k\left(D_q x_0 + \Delta\left(\frac{Q}{q}\right)D_q b \right) \\ &= \Delta(Q)_k(x_0) + \Delta\left(\frac{Q}{q}\right)_k\left(\Delta(Q)b\right) \quad \text{for } 1 \leq k \leq n, \end{aligned}$$

and

$$A_j = \prod_{i=k, i\neq j}^{n} \frac{Q_i}{Q_i - \frac{q_i}{q_j}Q_j} \quad \text{for } k \leq j \leq n.$$

Since $x(t) = D_{1/q}\, y(t)$, then for all $n \geq 1$, $x_n(t) = \frac{1}{q_n}y_n(t)$ and hence the unique solution of equation (4.1) is given by Formula (4.2). $\qquad\square$

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